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ABSTRACT

Awareness of the need for essential reforms within mathematics education evolved fundamentally as the consequence of several national reports, culminating in the documentation of this need with "Everybody Counts" in January 1989. The publication of "Curriculum and Evaluation Standards for School Mathematics" by the National Council of Teachers of Mathematics (NCTM) in March 1989 provided the direction for reform and established the vision of what mathematics education should encompass during the 1990s. This Model Learner Outcome document of the Minnesota Department of Education manifests that vision and is intended to assist schools in reviewing and updating their mathematics curriculum to reflect the expanding role of mathematics in everyday life. The document includes: (1) an enumeration of the educational values, the educational philosophy, and the tenets basic to the mission for public education of the Minnesota State Board of Education; (2) an historical perspective concerning the goals of mathematics education with emphasis on the components for reform and program-level learner outcomes; (3) an outline of essential concept-level learner outcomes that represents a common set of goals for full citizenship; (4) a structure of learner outcomes within the framework of the NCTM Standards arranged by the grade levels K-4, 5-8, and 9-12 for the development of a clear direction for the school mathematics curriculum as well as ideas on context and delivery within that curriculum; and (5) appendices including the components of mathematics education reform including the role of technology, the need for adequate teacher preparation, the quest for higher order thinking skills, and the levels and styles of individual cognitive development; a position report; an executive summary; mathematics contacts; and charts. (JJK)

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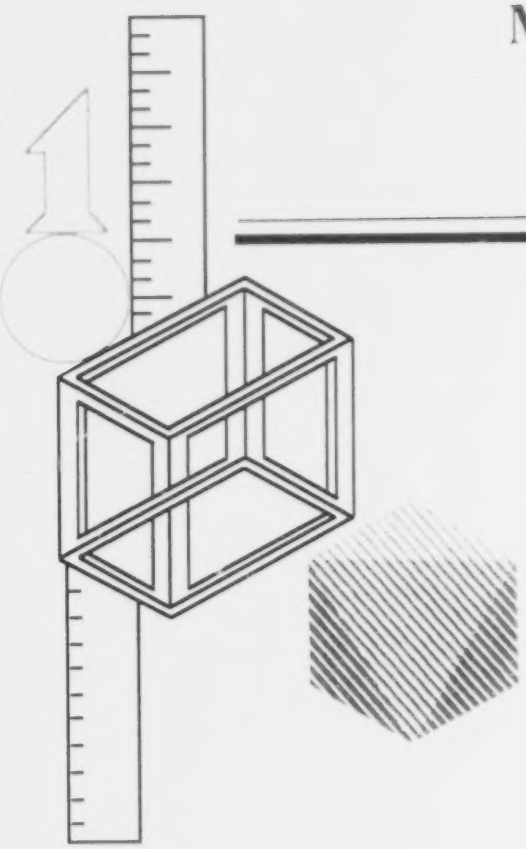
Model Learner Outcomes for Mathematics Education



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1991

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MODEL LEARNER OUTCOMES

FOR

MATHEMATICS EDUCATION

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1991 **Minnesota Department of Education**

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PREFACE

"'To some extent, everybody is a mathematician.' ...School mathematics must endow all students with a realization of this fundamental truth."
NCTM *Standards*

Since the publication of the *Model Learner Outcomes for Mathematics Education* in 1988, two notable events have occurred. They are the publication in April of 1989 of the *Curriculum and Evaluation Standards for School Mathematics* by the National Council of Teachers of Mathematics and the adoption in February 1990 of program level learner outcomes for mathematics by the State Board of Education.

These two events form the basis for this revision of the *Model Learner Outcomes for Mathematics Education.* The outcomes in Chapter 3 have been slightly revised and totally reorganized under the new program level learner outcomes but have not substantially changed. These outcomes correspond closely to the *Curriculum and Evaluation Standards for School Mathematics.* The new Chapter 4 is organized by the three grade levels used in the *Standards*: K-4, 5-8, and 9-12, and integrates the concepts of the learner outcomes with the *Standards.* The wording, which is largely from the *Standards,* incorporates all the model learner outcomes at appropriate grade levels. A local district could use either Chapter 3 or Chapter 4 as a basis for curriculum design and be confident that all important mathematical concepts would be included in its curriculum, including all the essential learner outcomes.

ACKNOWLEDGEMENTS

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Chapter 1

Mathematics Education

Minnesota State Board of Education

Education Values, Learner Values

Philosophy, Mission and Goals

Chapter 1

Minnesota State Board of Education

Values, Philosophy, Mission, and Goals

School districts nationwide, and certainly in Minnesota, are constantly striving to improve the learning experiences they provide students. The last two or three decades have seen heightened interest in improving all parts of the education process, including appropriate involvement of students, parents, and the community at large.

The documents that constitute the Minnesota Department of Education's *Coordinated Model for Educational Improvement* incorporate many of the concerns expressed and issues addressed by the public, Legislature, and in reports on the state of public education. One of these documents, titled *Goal and Outcome Specification Process*, suggests a set of procedures for appropriate involvement of the public. These procedures include public participation on the development of statements of values, philosophy, mission, and learner goals. These sets of statements are a hierarchy of increasingly specific concepts ranging from values, the most general, to learner goals, the most specific, that give form and direction to public education. Given this hierarchy, staff skilled in subject matter and the profession of teaching can develop very specific learner outcomes for each subject area.

The following sets of statements were adopted by either the Minnesota State Board of Education or the Minnesota State Legislature for two purposes. First, they provide a model for use by communities and school staff as they strive to improve the learning experiences they provide for residents. Second, they are the hierarchy used by Department staff and teams of educators as they develop model learner outcomes for each subject area. The *Mission Statement for Public Education* adopted by the Legislature gives explicit direction to public schools. The State Board Curriculum Rule 3500.1060 adopted February 1990 lists the learner goals which must be incorporated into each district's goal statements. All other parts of this document are models, suggestions for the consideration of residents and professionals in each district.

Education System Values

We believe the following values are preeminent for the education system. These attributes are to be reflected in all educational programs and operations.

Accountability - A condition in every school whereby each is able to justify its use of public resources by effectively fulfilling its mission of learning.

Effectiveness - A condition in every school whereby each accomplishes its mission at a performance level defined by learners, parents, citizens of the community and state, and their representatives.

Efficiency - A condition in every school whereby each accomplishes the highest possible level of excellence with available resources.

Excellence - A condition in every school whereby the highest possible standards for performance are expected of all students and staff.

Flexibility - A condition in every school which results in meeting the needs of learners through sensitive and creative responses to changing circumstances.

Human Equity - A condition in every school which offers equal opportunity and appropriate individualized support to each staff member in employment and professional growth and to each learner in the educational process. Also, a condition which fully, fairly, and accurately portrays various cultures, races, and genders in the instructional program.

Responsibility - A condition in which the school recognizes that the parent has primary responsibility to assure the child is educated, and in which a partnership exists between the school, community, parent, and the learner to identify the learning goals and needs of the child or adult learner and provide appropriate learning opportunities through which those goals can be met.

Responsiveness - A condition in every school whereby diversity of personal and group needs and aspirations are expected, accepted, encouraged, and routinely addressed.

System Equity - A condition in the education system whereby each school is provided with the resources necessary to assist all learners in achieving excellence.

Wholeness - A condition in every school whereby each gives necessary and appropriate consideration to the potential career needs, spiritual, social, emotional, and physical growth of each learner and staff member as it designs and implements educational programs.

Visionary - A condition in the education system whereby emerging trends which will affect the knowledge and skills required to be a successful adult are examined and the knowledge gained is used to produce appropriate changes in the system's course content, procedures, and goals for learners.

Learner Values

We believe helping students develop the following values is a primary purpose of education.

Accountability - A quality in individuals whereby each knows, understands, and accepts the impact and consequences of personal actions and decisions.

Citizenship - A quality in individuals whereby each has an understanding, appreciation, and support of the institutions of American government and society, and a willingness and ability to participate in the democratic process and in socially beneficial service activities.

Compassion - A quality in individuals whereby each is sensitive to the conditions affecting the lives of others and each has the commitment to assist others when appropriate and possible.

Competence - A quality in individuals whereby each attains maximum levels of knowledge, skill, and affect commensurate with his or her potential.

Cooperativeness - A quality in individuals whereby each interacts with others in a manner that mutually benefits all participants in the interaction.

Creativity/Flexibility - A quality in individuals whereby each acts or expresses self in new, improved, or unique ways.

Ethics - A quality in individuals whereby each displays consistent personal and professional integrity and an acceptance of the responsibility to act for the benefit of all learners.

Honesty - A quality in individuals whereby each is fair and straightforward in the conduct of human interaction.

Learning - A condition in individuals whereby each continually strives throughout life to learn more and to increase personal levels of fulfillment and competence in human endeavors.

Problem Solving - A condition in individuals whereby each has the ability to identify, frame, and propose new, improved, or unique solutions to existing and emerging problems.

Responsibility - A quality in individuals whereby each strives to fulfill the obligations of economic self-sufficiency and active commitment to the common good of society.

Self-Acceptance - A quality in individuals whereby each has a positive self-image, through assertion of rights, holding personal, physical, and emotional well-being as an ideal, accepting personal talents with humility, and personal limitations with the resolve to improve where possible and accept where necessary.

Spirituality - A quality in individuals whereby each recognizes and accepts the importance of nurturing one's inner spirit, that creative force that transcends the human and the material.

Thinking - A condition in individuals whereby each continually strives to improve personal skills for mental manipulation of sensory perceptions to form knowledge, thoughts, reason, and judgments.

Minnesota State Board of Education
Philosophy of Education

We Believe . . .

. . . Every person can learn.

. . . Learning is a lifelong process.

. . . Every person must understand and accept self before he or she can become a contributing member of society.

. . . Each person has gifts which the education program must seek, identify, and help to maximize.

. . . Advancement of the human race requires individuals who are honest, responsible, compassion- ate, cooperative, creative, and competent.

. . . The state and local communities have a shared responsibility to assist each person in learning.

. . . Each community has a shared responsibility with parents for meeting the needs of each child.

. . . The community and its school system must continually look to and strive to meet the future education needs of society.

. . . The education system must assist each person to become functional in an increasingly global and interdependent world.

. . . The education system must lead people to value and accept a wide diversity in human behaviors, sophistication, and values.

. . . The education system must maintain high standards for responsiveness, human equity, and system equity in the provision of educational opportunities.

. . . Decisions regarding planning, implementing, and maintaining learning opportunities must be vested as close to the individual learner as efficient use of public resources allow.

. . . School systems must implement programs that stress the intellectual development of each learner in concert with the spiritual, social, emotional, and physical develop- ment of the learner.

. . . School systems must model and nurture creativity in learners.

. . . School systems must maintain opportunities for all learners to develop competence in personally selected areas of human development.

. . . Professional educators have a responsibility to remain current with evolving knowledge about human growth and development, learning theory, and knowledge of subject matter.

. . . Professional educators and elected or appointed education policymakers have a responsibility to account to the public for the excellence of their efforts.

. . . Professional educators and elected or appointed education policymakers have a responsibility to use public resources prudently and efficiently.

. . . Staff employed in schools must model the behaviors they are assisting learners to develop.

Mission for Public Education

As adopted by theMinnesota Legislative Commission on Public Education and enacted into law, Chapter 240, Laws of 1985

The purpose of public education is to help individuals acquire knowledge, skills, and positive attitudes toward self and others that will enable them to solve problems, think creatively, continue learning, and develop maximum potential for leading productive, fulfilling lives in a complex and changing society.

Mission of the Minnesota State Board of Education

The Minnesota State Board of Education will provide the vision, advocacy, and leadership to improve significantly the quality of education throughout the state.

Mission of the Minnesota Department of Education

The Minnesota Department of Education provides leadership, service, and regulation to maintain and improve an equitable, uniform, and quality system of public education for all learners.

The Department provides leadership as an advocate for education by defining quality education and by seeking the resources necessary to meet the needs of all learners.

The Department provides service through informational and technical assistance that will improve the productivity and performance of students and staff, and provide opportunities for the development of the potential of all learners.

The Department regulates education by maintaining, interpreting, and enforcing Minnesota State Board of Education rules, and state and federal laws.

Minnesota State Board of Education

Learner Goals

Learner goals are a series of statements that describe the knowledge, skill, processes, values, and attitudes that a learner can expect to achieve as a result of active participation in K-12 public education. They are based on the presumed current and future intellectual, social, emotional, physical, and career/vocational needs of students and adults in contemporary society.

Each district shall use the learner goals contained below as the basis for defining program-level learner outcomes that are directly reflected in the district's course and program offerings.

A. To effectively participate in learning activities, each learner will:

1. master reading literacy to gather information and data, gain perspective and understanding, and as a leisure activity;
2. master writing to explain, describe, and express a point of view and feelings;
3. master listening to gather information and data and gain perspective and understanding;
4. master speaking to explain, describe, express a point of view and feelings, and to discuss an issue;
5. master numerical literacy to apply mathematical functions to life situations;
6. master the use of a variety of tools, including electronic technology, to enhance learning;
7. master viewing and observing to gather information and data, and gain perspective and understanding; and
8. apply skills in self-expression through visual and performing arts.

B. To provide a foundation for meaning in life, each learner will accumulate and apply knowledge and develop the understanding to:

1. participate in lifelong learning;
2. live within local, state, national, and world political and social structures;

3. examine personal beliefs and values and their relationship to behavior;

4. make ethical and moral decisions;

5. be a responsible citizen of the community, nation, and the world;

6. practice stewardship of the land, natural resources, and environment;

7. know the impact of human life on nature and the impact of natural phenomena on human life;

8. express self through artistic creation;

9. know career options and the general education requirements for each;

10. know world and national economic conditions to make informed decisions on consumer products, occupations and career needs, and use of resources;

11. select or prepare for a series of occupations that will personally satisfy and suit one's skills and interests;

12. manage personal affairs;

13. understand the physical world using systematic problem-solving strategies:

14. communicate and relate effectively in a language and about a culture other than one's own; and

15. know the importance of geographic location in the functioning of contemporary society.

C. To think, decide, resolve issues, and meet needs creatively, each learner will be able to:

1. compare, differentiate, and relate information and facts and apply knowledge;

2. combine various facts, situations, and theories to formulate new and original hypotheses or to develop new solutions;

3. critique and make judgments about materials, conditions, theories, and solutions;

4. generate and value creative alternatives; and

5. apply the concepts and processes of science.

D. To value, understand, and accept human interdependence, each learner will be able to:

1. seek interactions and feel comfortable with persons who are different in race, religion, social level, or personal attributes;

2. understand the basic interdependence of the biological and physical resources of the environment;

3. understand the interrelationships among complex organizations and agencies in modern society;

4. understand how the citizens of the United States are geographically and socially connected to people and places in other parts of the world.

E. To value, understand, and accept the diversity of humankind, each learner will be able to:

1. base actions and decisions on the knowledge that individuals differ in many ways;

2. base actions and decisions on the knowledge that values and behaviors differ from one social group to another;

3. base actions and decisions on the understanding that lifestyles and behaviors reflect the value system of the societies in which they were learned;

4. judge other's actions with an understanding of the personal and social context of that action;

5. accept that there is more than one way of being human;

6. base actions and decisions on the understanding that as individuals move from one society to another they can learn lifestyles and can learn to behave appropriately in different social contexts; and

7. act on the belief that human behavior is influenced by many factors and is best understood in terms of the context in which it occurred.

F. To address human problems through group effort, each learner will develop the knowledge, skills, values, and attitudes essential to:

1. act in accordance with a basic ethical framework incorporating the values that contribute to successful community life such as honesty, fairness, compassion, and integrity;

2. understand the importance of working in groups to achieve mutual goals; and

3. be able to provide leadership in resolving personal and societal issues.

G. Each learner will be able to effectively resolve conflicts with and among others by:

1. assuming responsibility to form productive and satisfying relationships with others based on respect, trust, cooperation, consideration, and caring for other persons;

2. acting on the belief that each individual has value as a human being and should be respected as a worthwhile person; and

3. resolving conflict in the manner most beneficial to society.

H. Each learner will be able to act on contemporary events and issues with a perspective of their historical origins:

1. understanding the origins, interrelationships, and effect of beliefs, values, and behavior patterns in world cultures;

2. understanding one's own culture and historical heritage through the literary, aesthetic, and scientific traditions of the past;

3. being familiar with the ideas that have inspired and influenced humankind; and

4. understanding the manner in which heritages and traditions of the past influence the direction and values of society.

I. Each learner will develop a positive attitude toward self, demonstrated through:

1. a feeling of positive self-worth, security and self-assurance;

2. a willingness to live with one's strengths and weaknesses;

3. a basic understanding of one's own body, its systems and physiology, and a positive attitude toward one's own physical appearance;

4. understanding that efforts to develop a better self contribute to the development of a better society;

5. understanding that self-concept is acquired by interaction with other people; and

6. appropriate control or release of emotions.

J. To set and achieve personal goals, each learner will develop the ability to:

1. select appropriate personal learning goals;

2. make decisions about one's life;

3. plan, act, and organize to realize one's goals;

4. accept responsibility for personal decisions and actions;

5. work now for goals to be realized in the future; and

6. select viable alternatives for actions in changing circumstances.

K. To cope with change, each learner will develop the ability to:

1. initiate appropriate change while respecting existing structures and concepts;

2. tolerate ambiguity;

3. understand that coping with change is a lifelong process;

4. understand and accept the changing nature of work and the potential need to change careers several times;

5. use career information and counseling services to make informed and satisfying vocational choices; and

6. understand that all knowledge is tentative and that as new discoveries are made the knowledge base grows.

L. To lead a healthy and fulfilling life, each learner will:

1. assume responsibility for one's own physical and mental health and safety by establishing a daily regime of health behaviors that will maintain mental and physical health and motor fitness;

2. make informed decisions about health products and services;

3. make a lifestyle that promotes healthful family living;

4. understand public health measures and their effect on the individual, family, community, and environment; and

5. be able to enjoy play-skill activities that include understanding, cooperation, accepting rules, controlling emotions, following group process, and acquiring self-satisfaction.

M. To lead a productive life and actively contribute to the economic well- being of our society, each learner will develop the work readiness skills of:

1. applying the basic skills of communications, computation, and scientific principles to real-life situations in a technological society;

2. defining and interpreting the nature of the work force in terms of one's own challenges and opportunities;

3. leadership and citizenship necessary to succeed as an active agent in a changing work force;

4. understanding employment opportunities, job seeking and keeping, and specific work as they relate to transition from school to economic productivity;

5. developing pride in good work and expecting quality in products and services; and

6. adopting a positive attitude toward work including the acceptance of the necessity of making a living and an appreciation of the value and dignity of work.

Chapter 2

Mathematics Education

Introduction

Goals

Program Level Learner Outcomes

Chapter 2

INTRODUCTION AND GOALS

OF

MATHEMATICS EDUCATION

The study of mathematics has existed at least as long as recorded history. Its influence on culture has been both profound and complex. Mathematics has changed immensely over the centuries. What was once considered to be a somewhat static field of knowledge has in the last few centuries taken on a dynamic status. Significant discoveries in the discipline of mathematics have occurred with regularity right up to the present decade; now the use of computers and calculators has accelerated new developments in mathematics. Extrapolating the rate of discoveries into the next few years should convince many that the next ten years will be an exciting and challenging time to be studying mathematics in Minnesota's elementary and secondary schools. The capability of computers and other related technologies to aid the study of mathematics also should enhance the excitement of mathematics instruction.

Since the study of mathematics is dynamic, it is advantageous to review and update the mathematics curriculum in Minnesota elementary and secondary schools at this time. A cursory review of the present mathematics curriculum reveals the need to look at significant changes in both the mathematics content and instructional delivery systems. Our present mathematics curriculum was designed to prepare shopkeepers, farmers, and factory workers for the 1940s. Admittedly, some of the skills needed to be a shopkeeper in 1943 are still useful today, but in the 1990s the demands of society for all citizens, male and female, of all cultures and backgrounds, well prepared in the area of mathematics, will far exceed the basic skills. Therefore, a revised mathematics curriculum must be specified, implemented, and evaluated in our schools.

HISTORICAL PERSPECTIVES

To understand why mathematics seems to be the focus of reform, one needs to go back at least 35 years and review what occurred from 1955 to the present. Any time 35 years of history are telescoped into a few lines, injustices and overgeneralizations may occur. Despite the pitfall, the following is a capsule view of mathematics education in five-year segments starting from 1955.

1955-1960

Although curriculum reform efforts were started earlier, a space shot of Sputnik by the Soviets propelled the United States to pour even more money and effort into mathematics curriculum revisions.

1960-1965

Historically, this period is usually labeled the "Modern Math Era." Textbook revisions of the 1960-1966 period based on experimental projects of the 1950s became commonplace in the schools. An assumption was that understanding could be developed through mathematical precision and rigor.

1965-1970

Immediately after the placement of the "modern math" textbooks, a backlash occurred: "Back to the Basics" became the watchword. Many schools hurried back to the pre-1960s textbook editions. During this time, the writings of Jean Piaget began to impact curriculum developments in mathematics, but mostly outside the United States. This period could be called, in a kind way, a transition period.

1970-1975

There was now broadening acceptance of Piaget's ideas in the United States. As a result, the curriculum was adjusted to include manipulative materials, lab materials, and hands-on activities. This period could be known as the lab mathematics period.

1975-1980

"Back to Basics" again reflected the general societal tone of the period from 1975-1980. Society wanted stability and mathematics was one area which was thought to offer stability. Unfortunately, no agreement was ever reached in what was basic or what we were going back to. Earlier movements highlighting the need for equal access in education for all students regardless of gender or race were being recognized to a greater extent. In addition, technological developments in society were making many basic skills almost obsolete.

1980-1985

The National Council of Teachers of Mathematics set the stage for this time period with its publication, *An Agenda For Action*. The council's number one recommendation was that "problem solving be the focus of school mathematics in the 1980s." Textbook publishers struggled to meet teacher expectations for problem-solving instruction. Problem-solving instruction was found to be more difficult and complicated to teach than anyone ever imagined.

1985-1990

This era of the realization of the need for reforms in mathematics education came about as a result of several national reports, culminating in the documentation of this need in *Everybody Counts* (January 1989). With the publication of the National Council of Teachers of Mathematics, *Curriculum And Evaluation Standards For School Mathematics* (March 1989) and *Reshaping School Mathematics*, Mathematics Science Education Board, 1990, the direction and vision for what mathematics education should look like for the 1990s was set. Not only the criteria for a quality mathematics curriculum from K-12 was designated, but attention was focused on how students come to understand mathematics.

Goals of Mathematics Education

This Model Learner Outcome document reflects the "vision" of the 1989-90 publications and is intended to assist schools in reviewing and updating their mathematics curriculum to reflect the expanded role of mathematics in everyday life. The Minnesota State Board of Education learner goals, as stated in State Board Rule and delineated in Chapter 1, expect that each student develop thinking and decision-making skills, therefore, the mathematics curriculum must expand its horizons and change its emphasis and strategies. A well-educated youth graduating from a Minnesota high school must know the fundamentals of mathematics, but more importantly, be prepared to be a contributing member of a pluralistic, information society, and a technological world.

Clearly, the task of the schools to prepare our youth to be effective problem solvers and decision makers is one not easily accomplished. Years of complacency have built up inertia and stagnation within the schools and within the mathematics curriculum, causing resistance to significant changes in how teaching and learning occurs.

The need to provide equal opportunities for females and minorities is broadening a teacher's perspective and expectations for all students. In addition, at the present time most of the mathematics taught is not determined by a school's curriculum guide or set of outcomes, but rather by the current textbook for the class. A major factor in changing the direction of the mathematics curriculum at any level, state, local, or classroom, is breaking the tradition of the textbook-driven or standardized test-driven curriculum. This frees us to view mathematics in a broader context. This document will be a guide which can assist schools in looking at mathematics in a new light and with a new critical view of what is essential for a well-educated youth in the state of Minnesota.

Mathematics Program Level Learner Outcomes

This *Model Learner Outcomes for Mathematics Education* document has been created to fulfill and support the accomplishments of the broad learner goals adopted by the Minnesota State Board of Education in February 1990. These are stated in Chapter 1. Those learner goals describe the knowledge, skill, processes, values, and attitudes that a learner can expect to achieve from active participation in K-12 public education. The learner goals are based on the mission for public education which was adopted by the Minnesota Legislature. They are further refined in consideration of the presumed current and future intellectual, social, emotional, physical, and career/vocational needs of contemporary students and adults. PER legislation requires each district to adopt a set of learner goals at the district level that incorporates the State Board of Education learner goals.

Program level learner outcomes define the scope and intended breadth of study in mathematics education. They represent the contribution that mathematics makes to the full range of learner goals. The mathematics program level learner outcomes appear in Minnesota Rule 3500.1075, Subp. 6, A-J restated below.

Opportunities will be provided for each student to learn to:

A. Apply mathematics in real-life problem solving;

B. Apply mathematics as a means of communicating ideas and relationships;

C. Apply mathematical reasoning to follow and judge logical arguments;

D. Value mathematics as an integrated, connected body of knowledge;

E. Apply computation and estimation, when appropriate;

F. Apply number relationships and functions;

G. Apply patterns, relationships, and number operations in everyday situations;

H. Understand geometric and spatial relationships that involve commonly encountered shapes and their uses;

I. Apply measurement, statistical description and analysis, and probablistic thinking to make decisions; and

J. Apply technology, when appropriate.

The achievement of the program level learner outcomes for mathematics education will assure both depth and breadth in sequential learning experience in mathematics.

Chapter 3 consists of the concept level learner outcomes for mathematics education. These outcomes are a listing of the primary learnings in mathematics and define the "body of knowledge" or "scope" of mathematics. These outcomes are not intended to be exhaustive, but rather models for individual schools in their curriculum review and development. They provide detail to the scope of knowledge, skill, and affect contributed by mathematics to the learner goals. In this revision, the concept outcomes have remained the same in content as the 1988 version of this document, but have been rearranged under the ten mathematical program level learner outcomes listed above. The essential learner outcomes have remained the same. Appendix E is a grid that correlates the new listing of outcomes with the 1988 version for reference.

Chapter 4 focuses on detailed assistance for teachers writing outcomes for more specific grade levels than described in Chapter 3. In particular, the outcomes are now enlarged, defined more clearly and separated into three categories: K-4, 5-8, and 9-12. This is to align the concept level learner outcomes with the recommendations of the National Council of Teachers of Mathematics *Curriculum And Evaluation Standards.*

Components of Mathematics Reform

During the research and development phase of the learner outcome process, some critical issues emerged and continued to be a focus of the discussions. These issues have also been included in recent reports calling for reform in mathematics education.

The following key issues were identified:

1. That the role of the calculator, computer, and other forms of technology should be more clearly understood and the power of technology used as a tool in mathematics instruction, as well as reshaping what is taught.

2. That all elementary teachers should have more training in mathematics, and there be teachers at the middle school level with more advanced training in mathematics and specialized roles.

3. That teaching higher order thinking skills be a focus in instructional strategies for all mathematics classes at every level.

4. That more responsibility should be taken for teaching the way children learn, more effort exerted into recognizing students' varied learning styles, instruction be developmentally appropriate, and more awareness of cultural and gender gaps present in traditional instructional patterns.

5. That strategies for teaching mathematics include:

   a. creating a classroom environment to support teaching and learning mathematics,
   b. setting goals and selecting or creating mathematical tasks to help students achieve these goals,
   c. stimulating and managing classroom discourse so that students and the teacher are clearer about what is being learned; and
   d. analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions.

6. That assessment in mathematics be properly aligned with the curriculum content and instructional techniques and that alternative methods of assessment be used.

Building on the last thirty years of mathematics reform and development, the six identified issues seem to complement and reinforce new directions for mathematics. Any one of them taken alone as the only direction would show a degree of shortsightedness. However, taking the six directions together may appear to result in an eclectic and wobbly collection. The ideal would obviously be to view the issues as objects progressing simultaneously. Mathematicians should have little difficulty viewing the development of issues simultaneously.

Further development and expansion of these issues appear in the Appendix.

Chapter 3

Mathematics Education

Concept Level Learner Outcomes

Essential Learner Outcomes

Chapter 3

CONCEPT LEVEL

MODEL LEARNER OUTCOMES FOR MATHEMATICS

Since mathematical thinking is an integral part of modern daily life, it is necessary that each student be exposed to all areas of mathematics to the greatest degree possible, and that each student be encouraged to maximize his or her potential in the subject. This list of outcomes represents a common set of goals toward which every person needs to work to be a full citizen living in a pluralistic society. Many students will extend their study far beyond the outcomes listed here, because of interest, career goals, encouragement from parents and teachers, and many other factors. Other students will address these common outcomes to varying degrees. The educational plan for all students should include as many outcomes as the ability of the student allows.

OUTCOME BASED EDUCATION

One of the basic elements of the Minnesota model for Outcome Based Education (OBE) supported by the Minnesota State Board of Education is learner outcomes. Minnesota documents list these outcomes in a hierarchy, from the most broad, general goals as listed in Chapter 1, to the program level learner outcomes referred to in Chapter 2, and to the more specific concept level learner outcomes offered as models in this chapter. The document *A Minnesota Vision For Outcome Based Education* (see bibliography) provides further information.

ORGANIZATION OF CHAPTER THREE

This chapter lists the model learner outcomes for mathematics education. While the outcomes themselves are substantially the same as those of the 1988 document, they are here arranged under the program level learner outcomes (see Chapter 2) as designated by the State Board of Education Curriculum Rules 3500.1075, Subp.6. (1990). A matrix in the Appendix correlates these outcomes with those in the 1988 document.

The format of the model learner outcomes includes the correlation with the State Board of Education Learner Goals (see Chapter 1) in the first column. Following the statement of the outcome is the identification of developmental or concept emphasis as it addresses knowing, applying, integrating, etc., as well as correlation with subject areas to be integrated into all courses, such as international perspectives, career and work readiness, and information technology.

It is not the intent of this document to be a fully prescriptive listing of "course curriculum." Districts need freedom and flexibility to adapt course content to local situations and resources. Rather, this is a listing of suggested outcomes that would become common to the mathematical background and experience of all students in Minnesota schools.

ESSENTIAL LEARNER OUTCOMES

Included in the model learner outcomes are the concept outcomes specified by the State Board of Education as Essential Learner Outcomes (ELOs). These outcomes are a **limited** sample of the mathematics outcomes and are indicated in this chapter by being underlined. The underlined essential learner outcomes range from low level rote memorized items to high level thinking, reading, and decision-making skills. The state has developed assessment procedures and has set standards for these outcomes. These procedures will be used by the state in its four-year testing cycle. This assessment is carried out on a sample basis, at three grade levels (grades 6, 9, and 11), to determine how well the state as a whole is performing in the essential outcomes. The procedures are given to a sample of students in a stratified-by-size sample of districts. According to Planning, Evaluation and Reporting (PER) legislation, these underlined essential learner outcomes must be included when the local district develops mathematics learner outcomes on its review cycle.

ADVANCED TOPICS

Many topics traditionally covered in high school are not included in these outcomes because of the variety of mathematics courses available in grades 9-12. Included after the concept outcomes in each section (after the XXX line) are those learner outcomes which districts provide opportunities for students that will prepare them for continuing their education beyond the secondary level. While not all of these topics are appropriate for all students, certain outcomes can be identified which are important for students planning to enroll in a postsecondary baccalaureate program. These concepts and skills are traditionally found in the mathematics offerings in grades 9 through 12. However, access to these topics should begin in grade eight or earlier. These advanced topics outcomes will be further developed in the 9-12 section of Chapter 4.

REQUIRED OFFERINGS

At the present time, in the transition period from clock hours or carnegie credits to graduation standards based on outcomes achieved, all students are required to take mathematics through grade nine. In addition, the Commissioner of Education, the State Board of Education, and the Legislature adopted a rule change (February 1990) which increased the minimum program **offerings** as follows:

(1) Grade 9, each district shall offer one, one credit course in algebra 1.

(2) Grades 9, 10, 11, and 12, each district shall offer one, one credit fundamental mathematics concepts course sufficient for students to make progress toward the mathematics program level learner outcomes specified in part 3500.1075, Subpart 6.

(3) Grades 10, 11, and 12, each district shall offer at least one, one credit course in each of algebra 1, geometry, advanced algebra, and one, one credit course in advanced topics such as probability and statistics, discrete mathematics, precalculus, or calculus. At least one of the courses shall include trigonometry. Districts may develop a scope and sequence, by any course titles, for a four-year curriculum in mathematics that integrates these and other mathematical concepts selected by the district.

(4) In addition, each district shall offer courses, programs, or services sufficient to assist students to meet the district's assurance-of-mastery criteria for mathematics.

Model Learner Outcomes for Mathematics Education

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
A.* The student will apply mathematics in real-life problem solving			
	Students will demonstrate that they:		
B.11, 13	A. can estimate, compare, and compute cost, (e.g., cost per unit, given the total cost and the number of units);	Applying Family life	IX. A.
B.11,13	B. can use and define the terms: gross, net, profit, loss, selling price, cost and discount, when solving problems;	Applying Career Work readiness	IX. B.
B.13	C. can count out an appropriate amount of money to pay for a purchase, given its cost, and estimate the correct change;	Knowing Career Work readiness	IX. C.
B.13	D. can determine specific consumer-related transactions: 1. the amount of sales tax and the total cost of a purchase 2. the amount of discount and sale price 3. comparison costs of credit buying and cash purchase	Applying Career Youth service	IX. D.
B.11, 13	E. can use unit cost comparisons to determine "best choices" for purchase of sizes and quantities;	Integrating Family life	IX. E.
B.11, 13	F. can discover fallacies in consumer advertising and propaganda, as well as data that might discriminate against gender, racial, or cultural differences;	Integrating Multicultural Career Family life	IX. F.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
B.11, 13	G. can solve consumer problems, including budgets, taxes, bank accounts, travel costs, arrangements, and schedules;	Applying Multicultural Career Family life Media Technology	IX. G.
B.13	H. can apply the content and techniques of mathematics in real-life situations;	Applying Career Youth service	IX. H.
C.1	I. can identify a problem-solving strategy from the following list: 1. guessing, checking, and refining 2. drawing a diagram or constructing a table, graph, or model 3. using patterns in analysis 4. selecting appropriate notation and formulating equations or other relational statements 5. working backwards 6. reducing to simpler cases 7. recursive thinking	Applying Career Environmental Family life Information Technology Media	XIII.A.
C.1	J. can demonstrate the application of a specific problem solving strategy (see listing in I);	Applying Career Environmental Family life Information Technology Media	XIII.B.
C.1,C.4	K. can select and use one or more mathematical strategies to solve problems;	Applying Integrating	XIII.C.
C.1	L. can determine relevancy of data and select appropriate data for solving a given problem;	Applying Information Technolgy Media	XIII.D.
F.2	M. can work cooperatively within a group (cross-cultural, cross-gender) as well as individually to formulate a process for solving a problem;	Affective/Applying Integrating Multicultural	XIII.F.
C.1	N. can complete a problem, including verifying and justifying the solution;	Applying	XIII.G.
C.4	O. can give more than one possible solution to an open-ended mathematical problem situation;	Integrating	XIII.H

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
C.3	P. can recognize how mathematics can be used to their disadvantage, e.g., misleading consumer ads, misuse of statistics, discrimination, etc.	Applying Multicultural Media	XV.C.
xxxxxxxxxxxxxxxxxxxxxxxxxxxxAdvanced Topics Outcomesxxxxxxxxxxxxxxxxxxxxxxxxxxxx			
	Students will demonstrate that they:		
C.1,2,3	AA. can understand and apply mathematical methods, steps, and processes;	Applying	XVII.A.
	BB. can create algorithmic procedures to solve problems;	Applying Integrating	XVII.C.
A, C.1,2,3, C.4, M.1	CC. can use mathematical concepts to define, describe, and represent real-life situations and the physical world;	Applying Integrating Youth service	XVIII.A.
C	DD. can use the methods and techniques of mathematics effectively to aid in the interpretation and solution of theoretical as well as real-life problems;	Application Integrating Youth service	XX.A.
C	EE. can refine and extend problem-solving techniques and strategies to both routine and non-routine problem types.	Integrating	XX.B.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
B.*	**The student will apply mathematics as a means of communicating ideas and relationships**		
	Students will demonstrate that they:		
C.2	A. can support a mathematical conclusion with logical reasoning through a written or oral justification;	Integrating Youth service	X.B.
A.2	B. can express mathematical ideas clearly in both written and oral form;	Applying Media	XII.A.
A.1	C. can read and understand scientific, technical, and mathematical material as found in textbooks, magazines, manuals, and reports.	Integrating Multicultural Media	XII.B.
xxxxxxxxxxxxxxxxxxxAdvanced Topics Outcomesxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx			
	Students will demonstrate that they:		
A.	AA. can translate between language and symbolic notation;	Applying	XVI.A.
A,C.1	BB. can recognize and use the power of abstractions and symbolism in mathematics.	Applying	XVI.B.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
C.*	**The student will apply mathematical reasoning to follow and judge logical arguments.**		
	Students will demonstrate that they:		
C.1	A. can use basic set relations and operations, including 1:1 correspondence, equivalence, subset, union, intersection, complement;	Knowing	X.A.
C.3	B. can use logical reasoning techniques to draw conclusions; e.g., set up an algorithm.	Applying Integrating Media	XIII.E.
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxAdvanced Topics Outcomesxxxxxxxxxxxxxxxxxxxxxxxxxxxx			
	Students will demonstrate that they		
C.1,2,3,	AA. can recognize similarities and differences among mathematical structures, systems, and concepts.	Applying Integrating	XVII.B.
C.1,2,3	BB. can demonstrate a conceptual understanding of the foundations of mathematics and mathematical systems;	Applying Integrating	XIX.A.
C.1,2	CC. can exhibit a working knowledge of the principles of logic, inductive and deductive reasoning, and the nature of proof.	Applying Integrating	XIX.B.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
D.*	**The student will value mathematics as an integrated, connected body of knowledge**		
	Students will demonstrate that they:		
B.9	A. can recognize mathematics in our world and pluralistic society, including the arts, music, architecture, and games;	Applying Affective Multicultural	XIV.A.
J.1,J.3 J.4, B.10	B. can recognize the importance and relevance of mathematics to each individual and to society, particularly to those individuals that traditionally have not participated in mathematics, such as women and minorities, e.g., that career choices are enlarged by a good mathematics background, use of computers in society is creating a growing need for math;	Knowing Applying Global Gender Fair Multicultural Career/work readiness Youth service	XIV.B.
B.9	C. can see beauty and aesthetics in mathematics;	Knowing Applying/Affective Multicultural	XIV.C.
H.2	D. can recognize the contribution of all cultures to math, e.g., Arabic numerals, Greek geometry, African measurement...(history of math);	Knowing Global/ International Multicultural	XIV.D.
H.2	E. can recognize the interrelatedness of mathematics topics;	Integrating	XIV.E.
B.1	F. can continue to learn mathematics beyond that which is merely required;	Affective Applying	XIV.F.
B.9	G. can experience excitement and joy at the accomplishment of a mathematical task;	Affective	XIV.G.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
B.10, B.12	H. can recognize the importance of mathematics in their daily lives and the increasing importance of math in any and all career choices they may make throughout their lives;	Knowing Career/work readiness Youth service	XIV.H.
B.10	I. can recognize that math is for everyone--every race, culture, gender and ability level, and that it is for lifelong living, not just making a living;	Knowing Multicultural Youth service	XIV.I.
B.12	J. can understand the social consequences of elitism and discrimination which results in having a large group of the population unskilled in mathematics, science, and technology.	Integrating Multicultural Gender Fair International	XV.D.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
E.*	**The student will apply computation and estimation, when appropriate**		
	Students will demonstrate that they:		
A.5	A. can recognize and use the following symbols: 1. + - x ÷ (operation) * / (computer operations) 2. = > < ≠ (relations) 3. π % √ (special symbols)	Knowing Information technology	I.A.
A.5	B. can recall the basic facts of addition, subtraction, multiplication, and division for 0-9;	Knowing	I.B.
A.5	C. can perform the basic computational operations with whole numbers, integers, and rational numbers (both fractional and decimal form) using each where appropriate: 1. mental computation; 2. calculators and computers; 3. paper and pencil;	Knowing Information technology	I.C.
A.5	D. can mentally multiply and divide by 10, 100, and 1000;	Knowing	I.D.
A.5	E. can perform addition and subtraction of fractions with denominators of 2, 3, 4, 5, 6, 8, 10, 12, and 16;	Knowing	I.E.
A.5	F. can perform multiplications with fractions with demoninators of 10 or less;	Knowing	I.F.
A.5	G. can solve percentage problems (both paper and pencil and calculator)	Applying Information technology	I.H.
A.5	H. can apply arithmetic operations to solve quantitative real-life problems;	Applying	I.K.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
A.5	I. can recognize the difference between estimated results and a computed result;	Knowing	III.A.
A.5	J. can round off whole numbers to the nearest 10, 100, 1000, and decimals to the nearest whole, tenth, hundredth, or thousandth;	Knowing	III.B.
A.5	K. can estimate the result when given "large scale" problems (population, transportation, etc.);	Applying Environmental Family life	III.C.
A.1, A.5	L. can determine whether or not a proposed answer is reasonable;	Applying	III.D.
A.1, A.5	M. can give an estimated answer to a mathematical problem;	Applying	III.E.
A.1, A.5	N. can determine for a given problem a suitable margin of error;	Integrating	III.F.
A.5	O. can recognize when to underestimate and when to overestimate;	Integrating	III.G.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
F.* The student will apply number relationships and functions			
	Students will demonstrate that they:		
A.5	A. can find equivalent numerals and percentages for given fractions, decimals, and ratios;	Knowing	I.G.
A.5	B. can compute with numbers in scientific notation;	Knowing	I.I.
A.5	C. can compute with numbers in exponential form;	Knowing	I.J.
A.5	D. can recognize the two forms of a rational number–decimal and fraction;	Knowing	II.A.
A.5	E. can identify what place value a digit in a particular position represents;	Knowing	II.B.
A.5	F. can read, write, and understand scientific notation; e.g., 3.78 x 10;	Knowing Information technology	II.D.
C.1	G. can describe and extend the concept of a function to solve problems, create patterns, and define functional relationships;	Applying Integrating	X.C.
C.1	H. can use a coordinate plane to represent functions or relations and to solve problems;	Applying Integrating	XI.E.
A.5	I. can recognize formulas, equations, variables, constants, expressions, and inequality;	Knowing	V.A.
	J. can evaluate a simple algebraic expression;	Applying	V.I.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
C.1	K. can determine, given a value for two variables, the third variable in the following two formulas: $a = bc$ or $a = b + c$	Applying	V.B.
C.1	L. can compare two ratios and generate sets of equivalent ratios, using numbers and/or variables;	Applying	V.C.
C.1	M. can solve for the missing value, given any three values in a proportion, using numbers and/or variables;	Applying	V.D.
C.1	N. can select and apply formulas to solve problems;	Knowing Youth service	V.F.
C.1	O. can graph linear equations;	Applying	XI.D.
	P. can use a coordinate plane to represent functions or relations and to solve problems;		XI.E
	Q. can solve problems using algorithmic (step-by-step) procedures.	Applying Integrating	XVII.B.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
G.*	**The student will apply patterns, relationships, and number operations in everyday situations**		
	Students will demonstrate that they:		
A.5	A. can identify and use examples of the terms: factor, multiple, percent, decimal, fraction, ratio, exponent, prime, composite;	Applying	II.C.
A.5	B. can arrange in numerical order a set of fractional numbers with denominators of 2, 3, 4, 5, 8, 10, 12, and 16;	Knowing	II.E.
A.5	C. can arrange a set of decimal numbers in numerical order;	Knowing	II.F.
C.1	D. can express a simple verbal or written description using mathematical symbols and variables;	Applying	V.G.
C.1	E. can represent and solve problem situations with simple linear equations, and inequalities;	Applying Youth service	V.E.
C.1	F. can identify and use the properties of numbers and operations in algebraic or computational exercises, e.g., commutative, associative, and distributive properties;	Integrating	V.H.
B.13	G. can locate points on a map locally, nationally, and globally, given ordered pairs;	Applying Multicultural Family life International Media	XI.A.
C.1	H. can demonstrate an understanding of number sense by comparing and/or arranging numbers from a given set (e.g., integers, fractions, decimals);	Applying	
B.1	I. can describe, extend, analyze, and apply a wide variety of patterns.	Applying	

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
H.*	**The student will understand geometric and spatial relationships that involve commonly encountered shapes and their uses**		
	Students will demonstrate that they:		
A.5	A. can demonstrate an understanding of: 1. line, segment, ray, and angle (acute, right, obtuse, complementary, supplementary) 2. parallel, perpendicular, intersecting horizontal and vertical lines 3. base and altitude of polygons, radius center, and diameter of circles 4. polygons and circles 5. important properties of polygons and circles, e.g., sum of angles in a triangle, opposite angles in a parallelogram are equal, etc. 6. three-dimensional figures and shapes, e.g., common polyhedrons;	Applying	IV.A.
A.5	B. can apply concepts of area, perimeter or circumference, and volume to appropriate measuring situations;	Applying Career/work readiness	IV.B.
C.1	C. can recognize the various types of symmetries of a figure	Knowing	IV.C.
C.1	D. can use concepts of congruence, similarity, and transformation to compare lengths, areas, and volumes of geometric shapes;	Integrating	IV.D.
C.1	E. can use angle and line relationships to make comparisons and draw conclusions (both two- and three-dimensional);	Integrating	IV.E.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
C.1	F. can use properties of shapes, angles and line relationships to draw models (both two- and three-dimensional);	Applying	IV.F.
	G. can use instruments of measurement to make drawings or identify an appropriate drawing in order to arrive at conclusions.	Integrating Media	IV.G.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
I.*	**The student will apply measurement, statistical descriptions and analysis, and probabilistic thinking to make decisions**		
	Students will demonstrate that they:		
B.13	A. can recognize appropriate use of the customary and metric units of length, capacity, mass or weight, and temperature;	Knowing	VI.A.
B.13	B. can recognize and give the value of all coins and paper bills in common United States currency transactions and the value relationships among them;	Knowing Career/work readiness Family life	VI.B.
B.13	C. can recognize the units of time and the value relationships among them;	Knowing Career/work readiness Family life	VI.C.
B.13	D. can relate powers of ten to the use of metric prefixes: kilo-, hecto-, deka-, deci-, centi-, and milli-;	Applying	VI.D.
B.13	E. can measure in customary and metric units: length, capacity, mass or weight, and temperature;	Applying Youth service	VI.E.
B.13	F. can use conversion relations in measurement within the system being used, but not necessarily from one system to another;	Applying Career/work readiness Family life Youth service	VI.F.
B.13	G. can use clocks and calendars from various cultures to solve passages of time problem;	Applying Multicultural International	VI.G.
C.1	H. can select and use appropriate tools and units from a variety of measurement systems to make defined measurements;	Applying Media	VI.H.
	G. can use instruments of measurement to make drawings or identify an appropriate drawing in order to arrive at conclusions;	Integrating Media	IV.G.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
	H. can demonstrate an understanding of the properties of two-and three-dimensional shapes;	Applying	
	I. can use properties of shapes, angles, and line relationships to draw models, make comparisons, and reach conclusions (both two-and three-dimensional);	Applying	
A.5.	I. can identify common use of the terms: odds, chance, random, probability;	Knowing	VII.A.
	J. can construct a variety of representations of a given set of data, such as tables, charts, graphs, box plots, etc.;	Applying Media Career/work readiness	VII.B.
A.5	K. can compute and apply measures of central tendency and dispersion;	Knowing Applying	VII.C.
A.5	L. can determine the probability (chance) of an event happening;	Knowing	VII.D.
C.1	M. can, given an appropriate problem or activity; 1. determine a question to be answered 2. decide what data must be obtained 3. differentiate between biased and unbiased data 4. devise a plan to obtain and record data 5. organize the data in presentable form 6. use data to draw conclusions 7. explain and justify the conclusions 8. reevaluate the conclusions when new data becomes available;	Integrating Multicultural Media	VII.E.
A.1	N. can recognize limitations of conclusions or discriminatory data drawn from statistics, graphs, and/or tables;	Integrating Multicultural Media	VII.F.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
B.13, C.1	O. can use maps or scale drawings to: 1. approximate measurements, and 2. determine distances, directions, locations, areas, lengths, etc.;	Applying Multicultural Media Family life	XI.B.
C.1	P. can read and interpret tables of data and various types of graphs;	Applying Multicultural Media	XI.C.

SBE LEARNER GOALS	LEARNER OUTCOMES	DEVELOPMENT AND CONCEPT EMPHASIS	TEST ITEM BANK
J.* The student will apply technology, when appropriate			
	Students will demonstrate that they:		
A.5	A. can perform the basic computational operations with whole numbers, integers, and rational numbers (both fractional and decimal form) using calculators and computers;	Knowing Information technology	I.C.
A.5	B. can find equivalent numerals and percentages for given fractions, decimals, and ratios using a calculator;	Knowing Information technology	I.G.
A.6	C. can recognize the limitations of computing devices, e.g., limits of number of digits, truncating and rounding errors, etc.	Knowing Information technology Media	VIII.A.
A.6	D. can use a calculator, including memory functions, to compute the four arithmetic operations, the square root function, and percent;	Knowing Psychomotor Information technology	VIII.B.
A.6	E. can follow the sequence of commands as illustrated by a flow chart;	Knowing Information technology Media	VIII.C.
A.6	F. can determine and perform the process to be carried out on a calculator or a computer to do the calculations needed in everyday situations;	Applying Psychomotor Information technology	VIII.D.
B.4 F.3	G. can accept the responsibility accompanying mathematical knowledge and the use of technology;	Integrating Information technology Family life	XV.A.
B.4 F.3	H. can accept the responsibility for ethical behavior in the use of mathematics and technology, e.g., use of statistics to sway opinions, discrimination against females and minorities, etc.;	Integrating Multicultural Gender fair Information technology Family life Youth service	XV.B.

* Taken from State Board of Education Curriculum Rules, 3500.1075 *Program Level Learner Outcomes,* Subp. 6. *Mathematics.*

Chapter 4

Mathematics Education

Transposing Outcomes into the Curriculum

Chapter 4

INTRODUCTION

Our society requires every person to have a firm grasp of mathematics, with understandings that enable each citizen to meet any challenge that comes, whether in the workplace, at the voting booth, or in her or his personal life. The outcomes in Chapter 3 spell out some of the pieces to this understanding, pieces every mathematically educated student should have in place. The national document that provides a framework for the mathematical understandings needed by every student in grades K - 12 is the *Curriculum and Evaluation Standards for School Mathematics* by the National Council of Teachers of Mathematics (NCTM). Chapter 4 integrates the concepts of the *Model Learner Outcomes for Mathematics Education* into the wording of the *Standards*. This will give Minnesota teachers and educators a clear direction for mathematics curriculum as well as ideas on context and delivery for that curriculum.

In Chapter 4 the learner outcomes are placed within the framework of the NCTM *Standards*, arranged as are the *Standards*, by the grade levels K-4, 5-8, and 9-12. Listed under each standard are the outcomes related to that standard. The first four standards are common to all three levels and form a focus that permeates all of the other standards. They are: mathematics as problem solving, communication, reasoning, and mathematical connections. They describe the process of learning mathematics as well as its content.

Throughout the curriculum, the expectations are that students will have access to calculators, computers and other technology, and be encouraged to use them when appropriate. Their use should be integrated throughout the mathematics curriculum, including assessments, rather than be taught as a special topic.

For each grade level grouping there are variations in the emphasis, determined by the developmental level of the students. In the K-4 section, an emphasis is placed on forming basic concepts through active learning and on viewing mathematics as a "sense-making" activity, rather than just a set of rules and procedures.

In the 5-8 section students need to continue their active learning, with hands-on models for new mathematical ideas rather than only rote learning. They should also experience a wider variety of mathematical topics.

The 9-12 section addresses the fact that students are becoming more abstract learners and developing a variety of goals. Though all students need to continue learning and doing mathematics throughout their high school career, there will be some differentiation of depth and breadth of topics.

ORGANIZATION OF CHAPTER 4

CURRICULUM STANDARDS

GRADES K-4	GRADES 5-8	GRADES 9-12
1. Mathematics as **PROBLEM SOLVING**	1. Mathematics as **PROBLEM SOLVING**	1. Mathematics as **PROBLEM SOLVING**
2. Mathematics as **COMMUNICATION**	2. Mathematics as **COMMUNICATION**	2. Mathematics as **COMMUNICATION**
3. Mathematics as **REASONING**	3. Mathematics as **REASONING**	3. Mathematics as **REASONING**
4. Mathematical **CONNECTIONS**	4. Mathematical **CONNECTIONS**	4. Mathematical **CONNECTIONS**
5. Estimation		5. Algebra
6. Number Sense and Numeration	5. Number Systems and Number Relationships	6. Functions
7. Concepts of Whole Number Operations	6. Number Systems and Number Theory	7. Geometry from a Synthetic Perspective
8. Whole Number Computation	7. Computation and Estimation	8. Geometry from a Algebraic Perspective
9. Geometry and Spatial Sense	8. Patterns and Functions	9. Trigonometry
10. Measurement	9. Algebra	10. Statistics
11. Statistics and Probability	10. Statistics	11. Probability
12. Fractions and Decimals	11. Probability	12. Discrete Mathematics
13. Patterns and Relationships	12. Geometry	13. Calculus (conceptual underpinnings)
	13. Measurement	14. Mathematical Structure

Curriculum and Evaluation Standards for School Mathematics, National Council of Teachers of Mathematics, 19 ?.

Grades K - 4

INTRODUCTION

At the K-4 level students should be actively involved in doing mathematics. Their experiences with mathematics should be developmentally appropriate and relate to real-world situations. Children should be exposed to open-ended problems where several approaches or solutions are possible. Their reasons for validating their approaches and solutions should be communicated to and discussed with classmates. Having children discuss and write about their experiences in mathematics is a good way to connect with children's development in the language arts. Connections should be made among topics in mathematics and between mathematics and the other subject areas in the elementary school curriculum.

Problem solving, reasoning, communication, and making connections are four processes that should permeate the K-12 mathematics curriculum. At the K-4 level, estimation may also be regarded as an important process woven into learning mathematical concepts. For each content area of the following Standards, beginning with Standard 6, the K-4 section includes Integration Charts showing examples of how the processes can be integrated with content areas. Curriculum developers at a district level may wish to create more charts of this type as they examine their district's objectives.

National Council of Teachers of Mathematics Standards . . . Grades K-4

1. **Mathematics as Problem Solving**
2. **Mathematics as Communication**
3. **Mathematics as Reasoning**
4. **Mathematical Connections**
5. **Estimation**
6. **Number Sense and Numeration**
7. **Concepts of Whole Number Operations**
8. **Whole Number Computation**
9. **Geometry and Spatial Sense**
10. **Measurement**
11. **Statistics and Probability**
12. **Fractions and Decimals**
13. **Patterns and Relationships**

STANDARD 1
Mathematics as Problem Solving

Problem solving provides a framework for the learning of most mathematical concepts and skills. Many of the best problem-solving situations grow naturally from the students' environment and are posed by both the student and the teacher. A variety of strategies and techniques can be used in the solution process. Open-ended problems are to be valued and encouraged.

In grades K-4, the study of mathematics should emphasize problem solving so that students can:

1. *use problem-solving approaches to investigate and unaerstand mathematics content*

2. *formulate problems from everyday and mathematical situations*

    - participating in instructional activities which include real problems with manipulative/laboratory/outdoor experiences and technology

3. *develop and apply problem-solving strategies to solve a wide variety of problems*
   including such strategies as

    - recognizing patterns
    - guessing and checking
    - working backwards
    - diagramming
    - simulating
    - simplifying a problem
    - thinking logically
    - estimation

4. *verify and interpret results with respect to the original problem*

    - determining reasonableness of answers

5. *acquire confidence in using mathematics meaningfully.*

STANDARD 2
Mathematics as Communication

Mathematics is a language which, when used early, often, and appropriately, enhances and expedites learning. Communication plays an important role in helping students construct links between their informal, intuitive notions, and the abstract language and symbolism of mathematics.

In grades K-4, the study of mathematics should include numerous opportunities for communication so that students can:

1. *relate physical materials, pictures, and diagrams to mathematical ideas*

2. *reflect on and clarify their thinking about mathematical ideas and situations*

    - communicating through writing and speaking their understandings of the relationships between and among mathematical concepts, procedures, and symbols
    - communicating at each stage of conceptual development
    - using concrete, pictorial, and abstract representations
    - communicating in every area of mathematics

3. *relate their everyday language to mathematical language and symbols*

4. *realize that representing, discussing, reading, writing, and listening are a vital part of learning and using mathematics.*

STANDARD 3
Mathematics as Reasoning

A classroom environment that values reasoning conveys to students the importance of critical thinking and establishes a spirit of inquiry. Students should be encouraged to present their ideas and justify their solutions, problem-solving processes, and thoughts in a variety of ways. Children and teachers need to know that being able to explain their thinking about a possible solution is as important as its answer.

In grades K-4, the study of mathematics should emphasize reasoning so that students can:

1. *draw logical conclusions about mathematics*
2. *use models, known facts, properties, and relationships to explain their thinking*
3. *justify their answers and solution processes*
4. *use patterns and relationships to analyze mathematical situations*
5. *believe that mathematics makes sense.*

STANDARD 4
Mathematical Connections

It is important that students connect ideas both among and within areas of mathematics, and between mathematics and other subjects. Students must recognize general principles relevant to several areas rather than learn too many isolated concepts and skills. Ideas should flow naturally from one lesson to another, extending over several days so that connections can be explored, discussed, and generalized. Extended exposure to integrated topics will give students a better chance of retaining the concepts and skills that are taught.

In grades K-4, the study of mathematics should include opportunities to make connections so that students can:

1. *link conceptual and procedural knowledge*
2. *relate various representations of concepts or procedures to one another*
3. *recognize relationships among different topics in mathematics*
4. *use mathematics in other curricular areas*
5. *use mathematics in their daily lives.*

STANDARD 5
Estimation

Estimation skills and understanding enhance the abilities of students to deal with everyday quantitative situations. Terms such as "about," "near," "closer to," "between," and "a little less than," illustrate that mathematics involves more than exactness.

In grades K-4, the curriculum should include estimation so students can:

1. *explore estimation strategies*
2. *recognize when an estimation is appropriate*
3. *determine the reasonableness of results*
4. *apply estimation in working with quantities, measurement, computation, and problem solving.*

STANDARD 6
Number Sense and Numeration

It is essential that students understand numbers so that they can make sense of number use in their everyday lives. Teachers should provide a variety of activities with physical materials before emphasizing work with symbols.

In grades K-4, the mathematics curriculum should include whole number concepts and skills so that students can:

1. *construct number meanings through real-world experiences and the use of physical materials*

2. *understand our numeration system by relating counting, grouping, and place value concepts*
   for example, representing 50 as

   - 5 tens
   - 2 twenty-fives
   - 4 tens and 10 ones

3. *develop number sense*
   including understanding the

   - cardinal and ordinal meanings of numbers
   - relative magnitude of numbers
   - relative effect of different operations on numbers
   - measurement of references for common objects and situations in their daily environment

4. *interpret the multiple uses of numbers encountered in the real world.*

INTEGRATION CHART FOR STANDARD 6
NUMBER SENSE AND NUMERATION

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:	Using tens and ones blocks, how many different ways can you show the number 10? 20? 30? 40? 50? Make a chart of the number of ways to show each number. Can you guess how many ways you might be able to show the number 100?
COMMUNICATION:	Using tens and ones blocks, secretly form a number on one side of a folder. Tell your partner how to use the blocks to form the number. Lift the folder and see if you match.
REASONING:	Here are two plans for getting your allowance this month. Use a calculator to explore the plans and then tell which plan you would pick and why you would pick that plan. Plan 1: 50 cents a day for each day of the month. Plan 2: One cent on the first day and then each day double the amount you got the day before (one cent on day one, 2 cents (2x1) on day two, 4 cents (2x2) on day three, 8 cents (2x4) on day four, etc.)
CONNECTIONS:	Go through a book of world facts and select 10 facts. Change three so they are unreasonable. Ask someone else to figure out which three you changed.
ESTIMATION:	Have three different size containers with jelly beans in them. Estimate how many jelly beans are in each. Take 20 jelly beans out of each container. Now revise your estimates.

STANDARD 7
Concepts of Whole Number Operations

Understanding the fundamental operations of addition, subtraction, multiplication, and division depends on instruction which focuses on concepts and relationships rather than computation. Informal problem-solving experiences prior to symbolic work with operations is important in helping students connect their real-life experiences, intuitions, common sense, and informal language skills to mathematics. Students with a good sense for whole number operations are able to apply them meaningfully and flexibly to computational skills.

In grades K-4, the mathematics curriculum should include concepts of addition, subtraction, multiplication, and division of whole numbers so that students can:

1. *develop meaning for the operations by modeling and discussing a rich variety of problem situations,* such as

   - acting out real-life or fantasy problems
   - using manipulative materials to model them
   - verbalizing a problem from objects or pictures

2. *relate the mathematical language and symbolism or operations to problem situations and informal language*

   - realizing that symbols are useful tools that help students describe real-life mathematical situations

3. *recognize that a wide variety of problem structures can be represented by a single operation*

4. *develop operation sense*

   - seeing relationships among numbers
   - exploring the effects of operations on numbers.

INTEGRATION CHART FOR STANDARD 7
CONCEPTS OF WHOLE NUMBER OPERATIONS

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:

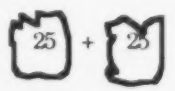
Leon brought this bag of candy to celebrate his birthday today. How could we make sure everyone in our class who wants some gets an equal number of pieces?

COMMUNICATION:

What information might we need to solve our problem with Leon's birthday treat?

REASONING:

Use your calculator to add the candies in each pair of bags.



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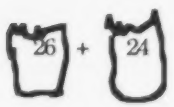

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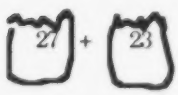

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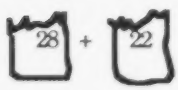


= _____

Now draw two more bags of candy that have the same sum as the sums above. How many different combinations of bags do you think you could draw? Explain.

CONNECTIONS:

Do you think this bag of candy costs the same at every store? Why? Why not? How could we find out? What different stores could we check? Who can check the price at Store A? Store B? etc. Make a table to show the data. What is the range of costs? How might these differences in price be explained?

ESTIMATION:

Estimate how many pieces of candy are in this bag. Estimate how many centicubes would balance this bag of candy on our balance scale. Discuss your estimate(s) with your partner and agree on an estimate between the two of you.

STANDARD 8
Whole Number Computation

Computational skills should enable students to develop a complete understanding of the four operations (+, -, x, ÷) and to apply these real-world problems.

In grades K-4, the mathematics curriculum should develop whole number computation so that students can:

1. *model, explain, and develop reasonable proficiency with basic facts and algorithms*
   including

   - activities with manipulative materials which precede the formulation of symbolic number facts and operations
   - development of algorithms with an understanding of underlying principles

2. *use a variety of mental computation and estimation techniques*
   so that they are able to

   - reach approximate solutions
   - judge the reasonableness of an answer or problem-solving strategy

3. *use calculators or computers in appropriate computational situations*

   - having both calculators and computers readily accessible in every classroom

4. *select and use computation techniques appropriate to specific problems and determine whether the results are reasonable*
   these techniques include

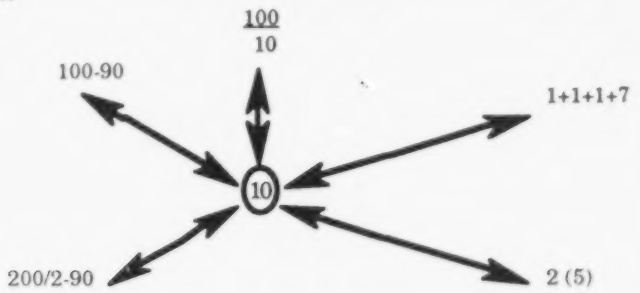
   - mental arithmetic
   - paper and pencil algorithms
   - calculating tools.

INTEGRATION CHART FOR STANDARD 8
WHOLE NUMBER COMPUTATION

An example of how to integrate the content of this standard with the processes of problem solving communication, reasoning, connections, and estimation.

PROBLEM SOLVING:

Using your calculator, find as many different ways of writing 10 as you can:



COMMUNICATION:

Draw a picture and write a "take away" story for this number sentence: 6 - 2 = 4

Now draw a picture and write a "comparison" story for this sentence: 6 - 2 = 4

Now draw a picture and write a "missing addend" story for this sentence: 6 - 2 = 4

REASONING:

George and his sister, Rosie, each spent 12 cents on gum at the store. They each gave the clerk a quarter.

George thought, "25 - 12 = _____" to figure his change. Rosie thought, "12 + _____ = 25" to figure how much change she should receive.

Who was right? Why?

CONNECTIONS:

Take 24 square tiles. Build as many different rectangles with them as you can. For each rectangle, write a multiplication number sentence that tells you about the area of each rectangle. How many different number sentences did you find? Explore this activity with different numbers of tiles.

ESTIMATION:

Estimate the answers to the questions below. Put your thumb up if you think the answer is more than twenty, thumbs down if you think the answer is less than twenty.

Examples: The group had 50 oranges. We ate 35. How many were left?
There were 15 people in the room. How many eyes are there altogether?

STANDARD 9
Geometry and Spatial Sense

Instructional activities involving spatial relationships and patterns enhance students' understanding and appreciation of our geometric world. The use of geometric terms should grow naturally from exploration and experience. Interest and success in mathematics can be enhanced by strengthening students' spatial abilities as well as numerical abilities.

In grades K-4, the mathematics curriculum should include two-and three dimensional geometry so that students can:

1. *describe, model, draw, and classify shapes*

2. *investigate and predict the results of combining, subdividing, and changing shapes*

    - by exploring solid and plane geometric shapes, attributes, and properties

3. *develop spatial sense*

    - gaining insights and intuitions into two- and three-dimensional shapes and their characteristics
    - discovering interrelationships of shapes
    - observing effects of changes on shapes
    - gaining ability to visualize real-world situations

4. *relate geometric ideas to number and measurement ideas*

5. *recognize and appreciate geometry in their world.*

INTEGRATION CHART FOR STANDARD 9
GEOMETRY

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:	Have students make a 1 foot by 1 foot square. Have them estimate and then prove how many of these squares would tile the floor in the classroom.
COMMUNICATION:	Have students work in groups and cut out objects with symmetry. Have the students prove an object is symmetrical by folding it in half. Each group will explain to the class how its object has symmetry.
REASONING:	Draw a design on one side of a line and have the students duplicate the design on the other side of the line so that it is symmetrical.
CONNECTIONS:	Have two or more students hold a long loop of yarn so each hand serves as a vertex of a polygon. Explore the effect on the shape of the polygon when: a) the size of an angle is changed, b) the number of sides is changed while the perimeter is unchanged, or c) the area is changed.
ESTIMATION:	Make a cylinder and rectangular prism out of equal-sized pieces of tagboard. Ask the students which container they think would hold the greatest amount. Have the students use a dry ingredient to check their answer.

STANDARD 10
Measurement

The process of measuring requires active involvement and encourages participation in problem solving. Experiences in measurement extend many mathematical concepts including number sense, estimation, proportional reasoning, and geometry. Many ways exist to integrate measurement across the elementary curriculum in real-life situations.

In grades K-4, the mathematics curriculum should include measurement so that students can:

1. *understand the attributes of length, capacity, weight, area, volume, time, temperature, and angle*

2. *develop the process of measuring and concepts related to units of measurement*

    - using customary and metric systems in real-life situations
    - developing non-standard systems of measurements

3. *make and use estimates of measurement*

4. *make and use measurements in problem and everyday situations.*

INTEGRATION CHART FOR STANDARD 10
MEASUREMENT

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:	How much floor covering would it take to cover the floor in this room? Would it be cheaper to buy squares of tiles or flooring by the square yard?
ESTIMATION:	How many gallons of paint would it take to paint the walls in this room? Why would you overestimate or underestimate?
COMMUNICATION:	Do different countries use different units of measurement? Does this cause any problems?
REASONING:	Take small steps and count the length of this room. How many large steps might it take?
CONNECTIONS:	How would you explain "five divided by one-half" as a measurement example using colored rods?

STANDARD 11
Statistics and Probability

Construction and interpretation of charts, tables, and graphs are important skills that should be developed. Students should translate real-life situations into graphical representations and use them to think analytically, categorize, compare, draw conclusions, and predict outcomes.

In grades K-4, the mathematics curriculum should include experiences with data analysis and probability so that students can:

1. *collect, organize, and describe data*
   - using real-life situations
   - using data from other content areas
2. *construct, read, and interpret displays of data*
3. *formulate and solve problems that involve collecting and analyzing data*
   - translating everyday real-life situations into mathematical representations
   - using representational models and/or verbal argument to justify thinking and solution processes
4. *explore concepts of chance*
   - using patterns, models, experiences, and observations to make and validate conjectures and/or predict outcomes.

INTEGRATION CHART FOR STANDARD 11
STATISTICS AND PROBABILITY

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING: Place 20 identical thumbtacks in a paper cup. Shake the cup and roll the tacks out on the desk. Count the number of tacks pointing up and record results in the table. Repeat several times. Ask students is "pointing up" more or less likely than "pointing down?" Why?

11	8				
9	12				

COMMUNICATIONS: Have students watch a news broadcast. Ask them to listen for and record any words they hear which are related to statistics, for example: average, monthly, daily, per person, etc.

REASONING: Make three boxes marked "certain," "uncertain," and "impossible." Make sentence cards such as "Our school will be struck by lightning." Have students place cards in appropriate boxes, and explain why. Extension: divide "uncertain" cards into piles marked "likely" and "unlikely."

CONNECTIONS: Do a survey on students' favorite fruit. Have them each bring their favorite fruit. Make a bar graph showing the favorite fruits. They could use their fruit to make a fruit bowl to share.

ESTIMATION: You have to buy bread for a week of school to make sandwiches for the entire class. Not everyone might like the same kind of bread. Estimate how much of each kind of bread you should buy. Estimate how many sandwiches you will be able to make from one loaf of bread. Make a graph of your results. Write a paragraph on how you made your estimations.

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STANDARD 12
Fractions and Decimals

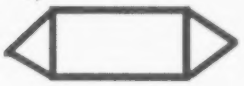
Students frequently experience fractions and decimals in their lives. These experiences can extend students' knowledge about number systems and provide an opportunity for developing initial concepts in fractions and decimals. A student's knowledge of number systems is greatly enhanced when fraction and decimal concepts are introduced early and integrated with whole numbers.

In grades K-4, the mathematics curriculum should include fractions and decimals so that the students can:

1. *develop concepts of fractions, mixed numbers, and decimals*
   - using physical materials, diagrams, and symbols
2. *develop number sense for fractions and decimals*
   including
   - relative size
   - relative effect of operating on fractions and decimals
   - referents for ordering and comparing
3. *use models to relate fractions to decimals and to find equivalent fractions*
   models such as
   - geometric and regular regions
   - sets
   - number lines
   - ratios
   - proportions
4. *use concrete models to explore operations on fractions and decimals*
   models such as
   - concrete materials
   - pictorial representations
5. *apply fractions and decimals to problem situations*
   - using real-world situations to show the relevance of fractions and decimals to daily life
   - applying concepts and operations to other areas of the mathematics curriculum such as geometry, measurement, probability, and statistics.

INTEGRATION CHART FOR STANDARD 12
FRACTIONS AND DECIMALS

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:	Shade in a sheet of graph paper using three different colors. What part of the design is color 1? Color 2? Color 3? What part is the combination of colors 1 and 3? 2 and 3? 1 and 2? Can you create other designs using the colors in the same proportions?
COMMUNICATIONS:	Use the estimation activity below. In small groups discuss ways used to find the estimate. Were any methods more successful than the others? Why?
REASONING:	Cut out the shapes below. Then cut each shape on the lines. Are all the parts for each shape equal? Which shapes have equal parts? 1) 2) 3) 4)  5) What other shapes can you make that have equal parts? What other shapes can you make that do not have equal parts?
CONNECTIONS:	Make cookies with the class. Have them measure out all ingredients. To extend the activity, make a half or doubled recipe.
ESTIMATION:	Estimate the size of each piece of cake for 25 people when the cake is 8 inches by 10 inches. NOTE: Bring in a small cake for the class and have them estimate the size of the piece each student will get.

STANDARD 13
Patterns and Relationships

Patterns are a natural part of students' experiences and can provide a foundation for studying more abstract mathematical ideas. Pattern activities should include work with concrete materials, diagrams, and symbols. Organizing and classifying information helps students see patterns and functional relationships. Patterns and relationships are critical in helping students see the structure, properties, and regularities of mathematics.

In grades K-4, the mathematics curriculum should include the study of patterns and relationships so that students can:

1. *recognize, describe, extend, and create a wide variety of patterns;*

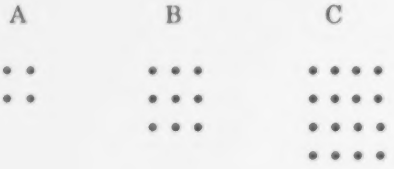
2. *represent and describe mathematical relationships*

   - representing a pattern in a variety of ways,
   - making connections among different representations of the same mathematical idea;

3. *explore the use of variable and open sentences to express relationships.*

INTEGRATION CHART FOR STANDARD 13
PATTERNS AND RELATIONSHIPS

An example of how to integrate the content of this standard with the processes of problem solving, communication, reasoning, connections, and estimation.

PROBLEM SOLVING:	How many squares can be drawn by connecting dots in each group? A B C  What are the different possibilities that happen in groups B and C? What do you think would happen if you had 5x5, 6x6, 7x7...arrays of dots?
COMMUNICATION:	How many squares can you form on a geoboard? How could you use different-colored bands to organize your thinking? How will you organize your work to tell others your results? How will you describe squares of different sizes?
REASONING:	If □ is a square unit on a geoboard can you make squares with area of 1,2,3,4,5,6,7,8,9,10 square units on the board? Explain your thinking.
CONNECTIONS:	Can you find designs in a wallpaper book that use squares? Do the designs use squares of the same size? Side-by-side squares? Squares within squares? Overlapping squares? How could you sort the designs? What other products do you think could use square designs? What kinds of square design are found in buildings?
ESTIMATION:	In the problem-solving section above estimate how many squares you can make by connecting the dots. Begin making squares and revise your estimate. Why did you revise your estimate?

Grades 5-8

INTRODUCTION

Mathematics is a useful, exciting, and creative area of study that can be appreciated and enjoyed by all students in grades 5-8. It helps them develop their ability to solve problems and reason logically. It offers to these curious, energetic students a way to explore and make sense of their world. However, many students view the current mathematics curriculum in grades 5-8 as irrelevant, dull, and routine. Instruction has emphasized computational facility at the expense of a broad, integrated view of mathematics and has reflected neither the vitality of the subject nor the characteristics of the students. Certain assumptions form the basis for the standards and outcomes listed:

- Every classroom will be equipped with ample sets of manipulative materials and supplies (e.g., spinners, cubes, tiles, geoboards, pattern blocks, scales, compasses, scissors, rulers, protractors, graph paper, grid-and-dot paper).
- Teachers and students will have access to appropriate resource materials from which to develop problems and ideas for explorations.
- All students will have a calculator with functions consistent with the tasks envisioned in this curriculum.
- Every classroom will have at least one computer available at all times for demonstration and student use. Additional computers should be available for individual, small-group, and whole-class use.

National Council of Teachers of Mathematics Standards...Grades 5-8

1. **Mathematics as Problem Solving**
2. **Mathematics as Communication**
3. **Mathematics as Reasoning**
4. **Mathematics Connections**
5. **Number and Number Relationships**
6. **Number Systems and Number Theory**
7. **Computation and Estimation**
8. **Patterns and Functions**
9. **Algebra**
10. **Statistics**
11. **Probability**
12. **Geometry**
13. **Measurement**

STANDARD 1
Mathematics as Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. It involves posing questions, analyzing situations, and translating and illustrating results. Students should see alternate solutions to problems and they should experience problems with more than a single solution.

In grades 5-8, the mathematics curriculum should include numerous and varied experiences with problem solving as a method of inquiry and application so that students can:

1. *use problem-solving approaches to investigate and understand mathematical content*

   - identifying problem-solving strategies
   - demonstrating the application of a specific problem-solving strategy
   - selecting and using one or more appropriate stategies in selected problem situations to solve the problem
   - looking for the interrelationships of mathematical ideas
   - generalizing specific problem situations to more complex or more general situations
   - changing one variable to observe what happens to other related variables in a problem situation
   - showing an understanding of mathematical concepts

2. *formulate problems from situations within and outside mathematics*

   - applying topics to real-world situations which may differ from those presented in the textbook
   - using computer simulations to model and/or analyze complex situations

3. *develop and apply a variety of strategies to solve problems with emphasis on multistep and non-routine problems*

   - collecting, organizing, and manipulating data and drawing inferences from that data
   - using mathematical information in various forms (e.g., tables, graphs, formulas, functions)
   - using several strategies or techniques in the solution process
   - giving more than one possible solution to an open-ended problem situation

4. *verify and interpret results with respect to the original problem situation*

   - formulating the problem
   - determining relevance of data and selecting appropriate data for solving a given problem
   - using logical reasoning techniques to draw conclusions, e.g., setting up an algorithm
   - completing a problem, including verifying and justifying the solution

5. *generalize solutions and strategies to new problem situations and other subject areas*

6. *acquire confidence in using mathematics meaning fully*

   - working cooperatively within a cross-cultural, cross gender group
   - working with problems that do not have easily processed procedures and that require perseverance
   - participating in class discussions to share problem-solving strategies.

STANDARD 2
Mathematics as Communication

The mathematics program should provide opportunities for students to develop and use the language and notation of mathematics. The communication process requires students to reach agreement about the meanings of words and to recognize the crucial importance of shared definitions with other disciplines. Mathematical ideas should be expressed by writing, speaking, and making models. Opportunities should be provided for mathematical discussions.

In grades 5-8, the study of mathematics should include opportunities to communicate so that students can:

1. *model situations using oral, written, concrete, pictorial, graphical, and algebraic methods*

    - writing mathematical problems from real-world situations
    - demonstrating appropriate mathematical symbolism

2. *reflect on and clarify their own thinking about mathematical ideas and situations*

    - evaluating ideas, and sharing solution strategies
    - verbalizing and writing about the problem-solving process used in reaching a solution

Small group cooperative learning should be used to encourage students to ask questions and discuss ideas.

3. *develop common understandings of mathematical ideas, including the role of definitions*

    - discussing the relationships between concepts and symbols
    - describing and representing concepts from concrete to symbolic notation

The ability to read mathematics should be emphasized.

4. *use the skills of reading, listening, and viewing to interpret and evaluate mathematical ideas*

   - listening to and questioning the ideas of others
   - developing the ability to read mathematics

Encourage students to take responsibility for validating their own thinking.

5. *discuss mathematical ideas and make conjectures and convincing arguments*

   - defending positions on mathematical processes and/or solutions through sound arguments
   - encouraging students to take responsibility for validating their own thinking

6. *appreciate the value of mathematical notation and its role in the development of mathematical ideas*

   - using proper mathematical vocabulary and notation
   - connecting experiences and language to the mathematical world.

STANDARD 3
Mathematics as Reasoning

The acquisition of reasoning skills should be emphasized in all mathematical activities in order to give more students access to mathematics as a powerful way of making sense of the world. Students need a great deal of time and many experiences to develop their abilities to construct valid arguments in problem settings and to evaluate the arguments of others. Both inductive and deductive reasoning come into play as students make conjectures and seek to explain why they are valid. Whether encouraged by technology or by challenging mathematical situations posed in the classroom, the freedom to explore, conjecture, validate, and to convince others is critical to the development of mathematical reasoning in the middle grades.

In grades 5-8, reasoning shall permeate the mathematics curriculum so that students can:

1. *recognize and apply deductive and inductive reasoning*

   - discussing the type of reasoning necessary to solve a problem situation
   - generating a set of specific instances, organizing them and looking for a pattern that can be generalized into a conjecture to be validated
   - posing specific problems that use a general mathematical statement, theorem, or concept

2. *understand and apply reasoning processes, with special attention to spatial reasoning and reasoning with proportions and graphics*

   - creating geometric and numeric situations to explore logical and proportional reasoning
   - devising and using scale models
   - investigating problems showing the relationship between graphical and algebraic relationships

3. *make and evaluate mathematical conjectures and arguments*
   - describing objects or processes accurately and elaborating their properties, similarities, differences, and relationships

4. *validate their own thinking*

   - explaining their reasoning in their own words, either orally or in writing

Listening to peers and their teacher describe other strategies will help students refine their own thoughts and the language they use to express their thoughts.

5. *appreciate the pervasive use and power of reasoning as a part of mathematics*

   - evaluating problems that have no solution
   - using counterexamples as a useful problem-solving strategy.

STANDARD 4
Mathematical Connections

Students' understanding of mathematical concepts should be expanded and deepened through the exploration of interconnections among mathematical ideas. Students need to have many opportunities to observe the interaction of mathematics with other school subjects and with everyday society. This integration of mathematics into contexts will give mathematical symbols and processes practical meaning. Students will experience how mathematics can help them understand others as well as how to use mathematics to describe and model real-world phenomena. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view.

Problem solving is used to foster an informal familiarity of the connections among mathematical topics.

In grades 5-8, the mathematics curriculum should include the investigation of mathematical connections so that students can:

1. *see mathematics as an integrated whole*

   - recognizing and drawing connections among mathematical topics
   - using problem solving to achieve an informal familiarity with the connections among mathematical topics
   - drawing on previously developed ideas
   - using technology as an integral part of the mathematics

2. *explore problems and describe results using graphical, numerical, physical, algebraic, and verbal mathematical models or representations*

   - exploring various ways to model mathematical relationships
   - using physical models as a problem-solving strategy that leads to algebraic and verbal representation of the problem

3. *use a mathematical idea to further their understanding of other mathematical ideas*

- exploring problem situations that involve several mathematical topics such as estimation, measurement, geometry, and patterns
- generalizing results to other situations and subject matter areas

4. *apply mathematical thinking and modeling to solve problems that arise in other disciplines, such as art, music, psychology, science, and business*

- using statistics to provide insights into economic and social problems
- connecting the context of mathematical problems to topics being discussed in other content areas
- designing interdisciplinary projects

Activities should instill in students an expectation that the ideas they learn are useful in solving other problems and exploring other mathematical concepts.

5. *value the role of mathematics in our culture and society*

- inquiring into and investigating real-world problems
- developing an awareness of the many relationships between formal mathematics and the real world
- working on projects based upon real-world problems that require extended time to complete.

STANDARD 5
Number and Number Relationships

In the middle years, students need to develop a lasting sense of number and number relationships. Learning should be grounded in experiences related to everyday life by using concrete materials. Students come to recognize that numbers have multiple representations. Development of concepts for fractions, ratios, decimals, and percents need special attention and emphasis.

In grades 5-8, the mathematics curriculum should include the continued development of number and number relationships so that students can:

1. *understand, represent, and use numbers in a variety of equivalent forms (integer, fraction, decimal, percent, exponential, and scientific notation) in real-world and mathematical problem situations*

    - understanding that numbers have several representations, such as fractions, decimals, percents, square root
    - writing numbers as numerals, in words, and in expanded notation
    - reading, using, and appreciating multiple representations of the same quantity
    - interpreting numbers that appear on calculators and computers including scientific notation
    - computing with numbers in scientific notation and exponential form
    - recognizing the difficulties inherent in various representations, (e.g., 1/3 as a percent of .999... = 1)
    - recognizing the limitations of computing devices, e.g., limits of number of digits, truncating, and rounding errors

2. *develop number sense for whole numbers, fractions, decimals, integers, rational, and irrational numbers*

    - discussing why a particular representation is useful in a given setting, e.g., why it is better to write 68/100 dollars on a check than to reduce to 17/25 dollars

3. *understand and apply ratios, proportions, and percents in a wide variety of situations*

   - using real-world problems that involve ratio and proportion, both direct and indirect
   - working with problems that include concrete, pictorial, and abstract representations

4. *investigate relationships among fractions, decimals, and percents*

   - developing processes to convert between fractions, decimals, and percents
   - presenting and using appropriately numbers with terminating or repeating decimal forms

5. *represent numerical relationships in one- and two-dimensional graphs*

   - understanding the relationship between a number or set of numbers and its (their) graph

   - using number lines, area models, graphs, and Cartesian coordinate systems.

STANDARD 6
Number Systems and Number Theory

Students should come to understand and appreciate mathematics as a coherent body of knowledge rather than a vast, perhaps bewildering, collection of isolated facts and rules. Understanding this structure promotes students' efficiency in investigating the arithmetic of fractions, decimals, integers, and rationals through the unity of common ideas. It also offers insights into how the whole number system is extended to the rational number system and beyond. It improves problem-solving capability by providing a better perspective on arithmetic operations. Number theory offers many rich opportunities for explorations that are interesting, enjoyable, and useful. These explorations have payoffs in problem solving, in understanding and developing other mathematical concepts, in illustrating the beauty of mathematics, and in understanding the human aspects of the historical development of numbers.

In grades 5-8, the mathematics curriculum should include the study of number systems and number theory so that students can:

1. *understand and appreciate the need for numbers beyond the whole numbers*

    - using the most appropriate form of a number
    - viewing mathematics as a systematic development of a body of knowledge
    - recognizing that numbers such as $\sqrt{2}$ and π are necessary to describe the length of the diagonal of a certain square and the circumference of any circle

2. *develop and use order relations for whole numbers, fractions, decimals, integers, and rational numbers*

    - arranging in numerical order a set of fractional numbers with denominators of 2, 3, 4, 5, 8, 10, 12, and 16
    - arranging a set of decimal numbers in numerical order
    - understanding that comparison of negative integers is different than comparison of positive integers, i.e., absolute value
    - using the number line is regularly used to compare numbers

3. *extend their understanding of whole number operations to fractions, decimals, integers, and rational numbers*

   - developing binary operations (addition, subtraction, multiplication, and division) with fractions, decimals, integers, and rational numbers from existing operations with whole numbers
   - using concrete materials and representational models to facilitate the transition from whole numbers to fractions, decimals, integers, and rational numbers

4. *understand how the basic arithmetic operations are related to one another in whole numbers, fractions, and decimals*

   - recognizing that subtraction can be considered in terms of addition, and that division can be considered in terms of multiplication (inverse operations)
   - recognizing that multiplication can be considered as repeated addition and division can be considered as repeated subtraction

5. develop and apply number theory concepts (e.g., primes, factors, and multiples) in real-world and mathematical problem situations

   - investigating other interesting topics such as abundant, deficient and perfect numbers, triangular and square numbers, cubes, palindromes, factorials, and Fibonacci numbers
   - exploring informally the relation of number theory to geometry.

STANDARD 7
Computation and Estimation

Students should be able to select and use appropriate mental, paper and pencil, calculator, and computer methods. It is no longer necessary or useful to devote large portions of instructional time to performing routine computations by hand. The facility in computation that calculators and computers offer should permeate the curriculum with an expanded potential for interesting problem solving experiences. Students need more experience in developing procedures and evaluating their work and in interpreting the results of computations done by machines. Computation, estimation, or methods for solving proportions should not be considered or taught as ends in themselves.

In grades 5-8, the mathematics curriculum should develop the concepts underlying computation and estimation in various contexts so that students can:

1. *compute with whole number, fractions, decimals, integers, and rational numbers*

    - using paper and pencil algorithms less in favor of exploring more mathematics
    - becoming proficient with simple calculations
    - using a wide variety of computation and estimation techniques suited to different mathematical settings
    - checking the reasonableness of computations, using number and algorithmic sense estimation skills

2. *develop, analyze, and explain procedures for computation and techniques for estimation*

    - counting out an appropriate amount of money to pay for a purchase, given its cost, and estimate the correct change
    - determining specific consumer-related transactions, e.g.,
        * the amount of sales tax and the total cost of a purchase
        * the amount of discount and sale price
        * comparison costs of credit buying and cash purchase
    - using unit cost comparisons to determine "best choices" for purchase of sizes and quantities
    - creating and posing problems for peers and teacher

3. *use estimation to check reasonableness of results*

- verbalizing why an answer or approach to a problem is reasonable
- recognizing the difference between estimated results and a computed result
- determining for a given problem a suitable margin of error and choosing an appropriate approximation method to estimate the solution
- analyzing the way various algorithms work and how they relate to the meaning of the operation and to the numbers involved
- discussing, comparing, and evaluating algorithms with peers and teacher

4. *develop, analyze, and explain methods for solving proportions*

- using informal methods for solving proportions
- using the concept of ratio and equivalent ratios including the unit ratio and relating percents to proportions
- using the cross multiplication method after more informal methods are understood

5. *select and use an appropriate method for computing from among mental arithmetic, paper and pencil, calculator, and computer methods*

6. *use computation, estimation, and proportions to solve problems*

- using a calculator, including memory functions, to compute the four arithmetic operations, the square root function and percent
- using concrete materials to solve problems
- solving percentage problems
- using a variety of strategies to solve routine and non-routine problems
- estimating, comparing, and computing - cost, e.g., find the cost per unit given the total cost and the number of units
- using and defining the terms: gross, net, profit, loss, selling price, cost, and discount in problem solving.

STANDARD 8
Patterns and Functions

One of the central themes of mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and build mathematical models to predict the behavior of real-world phenomena that exhibit the observed pattern. The widespread occurrence of regular and chaotic pattern behavior makes the study of patterns and functions important. Exploring patterns helps students develop mathematical power and instills in them an appreciation for the beauty of mathematics.

In grades 5-8, the mathematics curriculum should include exploration of patterns and functions so that students can:

1. *describe, extend, analyze, and create a wide variety of patterns*
    - exploring patterns as informal investigations by observing and describing all sorts of patterns
    - describing or analyzing patterns in various ways such as tables of data, graphs, and algebraic expressions
    - creating patterns to share with peers and teacher
2. *describe and represent relationships with tables, graphs, and rules*
3. *analyze functional relationships to explain how a change in one quantity results in a change in another*
    - describing and extending the concept of a function to solve problems
    - creating patterns and defining functional relationships
4. *use patterns and functions to represent and solve problems.*

STANDARD 9
Algebra

It is essential that in grades 5-8 students explore algebraic concepts in an informal way to build a foundation for the subsequent formal study of algebra. Such informal explorations should emphasize physical models, data, graphs, and other mathematical representations rather than facility with formal algebraic manipulation. Students should be taught to generalize number patterns to model, represent, or describe observed physical patterns, regularities, and problems. These informal explorations of algebraic concepts should help students to gain confidence in their ability to abstract relationships from contextual information and use a variety of representations to describe those relationships.

Activities in grades 5-8 should build on students' K-4 experiences with patterns. They should continue to emphasize concrete situations that allow students to investigate patterns in number sequences, make predictions, and formulate verbal rules to describe patterns. Learning to recognize patterns and regularities in mathematics and make generalizations about them requires practice and experience. Expanding the amount of time that students have to make this transition to more abstract ways of thinking increases their chances of success. Through informal algebraic experiences throughout the K-8 curriculum, students will develop confidence in using algebra to represent and solve problems. In addition, by the end of the eighth grade, students should be able to solve linear equations by formal methods and some nonlinear equations by informal means.

In grades 5-8, the mathematics curriculum should include explorations of algebraic concepts and processes so that students can:

1. *understand the concepts of variable, expression, and equation*

    - exploring the concepts of variable, expression, and equation using concrete materials
    - recognizing formulas, equations, variables, constants, expressions, exponents, and inequality
    - determining, given a value for two variables, the third variable in the following two formulas:
      $a = bc$
      $a = b+c$

- comparing two ratios and generating sets of equivalent ratios, using numbers and/or variables
- solving for the missing value, given any three values in a proportion, using numbers and/or variables

2. *represent situations and number patterns with tables, graphs, verbal rules, or equations, and explore the interrelationships of these representations*

   - exploring the connections and/or interrelationships between the various techniques for representing data

Encourage and help students to reflect on their explorations and summarize concepts, relationships, processes, and facts that have emerged from their discussions.

3. *analyze tables and graphs to identify properties and relationships*

   - identifying and using the properties of numbers and operations in algebraic or computational exercises, e.g, commutative, associative, and distributive properties
   - graphing linear equations
   - using a coordinate plane to represent functions or relations and to solve problems, e.g., distance, midpoint, slope, direction, angle determination, maximum, minimum

4. *develop confidence in solving linear equations using concrete, informal, and formal methods*

   - exploring linear equations from a concrete view using ideas such as a balance scale
   - exploring computer applications that help to visualize the concept of a linear equation
   - solving simple linear equations, inequalities, quadratic equations, and systems of linear equations
   - selecting and applying formulas to solve problems
   - expressing a simple verbal or written description using mathematical symbols and variables

As students become more confident solving linear equations using concrete materials, extend their understanding by providing parallel symbolic action.

5. *investigate inequalities and nonlinear equations informally*

   - using calculators and computers to investigate inequalities and nonlinear equations

6. *apply algebraic methods to solve a variety of real-world and mathematical problems*

Use small group cooperative activities to extend student's problem-solving strategies such as guess and check to more formal algebraic methods.

STANDARD 10
Statistics

In this age of information and technology, an ever-increasing need exists to understand how information is processed and translated into usable knowledge. Because of society's expanding use of data for prediction and decision making, it is important that students develop an understanding of the concepts and processes used in analyzing data. A knowledge of statistics is necessary if students are to become intelligent consumers, workers, and voters who can make critical and informed decisions. Students need to be actively involved in each of the steps that comprise statistics, formulating a question, gathering information, and communicating results.

In grades 5-8, the mathematics curriculum should include explorations of statistics in real-world situations so that students can:

1. *systematically collect, organize, and describe data*

   - gathering, organizing, and studying interesting and relevant data
   - discussing rationale for using sampling techniques
   - exploring a variety of techniques for organizing data
   - describing organized data both verbally and in paragraph form

2. *construct, read, and interpret tables, charts, and graphs*

   - using computer software to enhance the organization and representation of data
   - reading and interpreting tables of data and various types of graphs
   - using tables, charts, and graphs from a variety of everyday sources such as a newspaper to provide a focus for classroom discussion
   - constructing a variety of representations of a given set of data, such as tables, charts, graphs, box plots
   - creating and constructing graphs using computer applications when appropriate

STANDARD 11
Probability

Probability theory is the underpinnings of the modern world. Current research in both the physical and social sciences cannot be understood without it. Today's politics, tommorrow's weather report, and next week's satellites all depend on it.
(Huff and Geise 1959)

An understanding of probability and the related area of statistics is essential to being an informed citizen. Often we read statements such as: "There is a 20 percent chance of rain or snow today"; "The odds are three to two that the Cats will win the championship"; "The probability of winning the grand prize in the state lottery is 1 in 7 240 000". Students in the middle grades have an intense interest in the notions of fairness and the chances of winning games. The study of probability develops concepts and methods for investigating such situations. These methods allow students to make predictions when uncertainty exists and to make sense of claims that they see and hear.

In grades 5-8, the mathematics curriculum should include explorations of probability in real-world situations so that students can:

1. *model situations by devising and carrying out experiments or simulations to determine probabilities*

    - identifying common use of the terms: odds, chance, random, probability
    - using computers to simulate probability experiments
    - using the results of their own experiments to develop an understanding of the relationship between the numerical expression of a probability and the events that give rise to these numbers

Organize groups that actively involve students in probability experiments.

2. *model situations by constructing a sample space to determine probabilities*

   - constructing a sample space and determining the probability (chance) of an event happening, e.g., the occurrence of a head when flipping a coin, the occurrence of drawing a certain object from a given set

3. *appreciate the power of using a probability model by comparing experimental results with mathematical expectations*

   - guessing at the results of probability experiments before proceeding with the experiment and discussing the rationale for their guess
   - conducting experiments that can be easily compared to mathematical expectations in order to see connections between experimental and theoretical results
   - understanding that some probability problems do not have theoretical solutions

4. *make predictions that are based on experimental or theoretical probabilities*

   - understanding how predictions can be based on data in modeling problems, conducting simulations, and collecting, graphing, and studying data
   - using the results of experiments to model situations or predict events
   - analyzing theoretical probabilities and make predictions based on mathematical knowledge

5. *develop an appreciation for the pervasive use of probability in the real world*

   - reinforcing fractions, decimals, and percents through the study of probability
   - exploring events and situations relevant to daily lives.

STANDARD 12
Geometry

Geometry is grasping space... that space in which the child lives, breathes, and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe, and move better in it.
(Freudenthal 1973)

The study of geometry helps students represent and make sense of the world. Geometric models provide a perspective from which students can analyze and solve problems, and geometric interpretations can help make an abstract (symbolic) representation more easily understood. Many ideas about number and measurement arise from attempts to quantify real-world objects that can be viewed geometrically. For example, the use of area models provides an interpretation for much of the arithmetic of decimals, fractions, ratios, proportions, and percents.

In grades 5-8, the mathematics curriculum should include the study of the geometry of one, two, and three dimensions in a variety of situations so that students can:

1. *identify, describe, compare, and classify geometric figures*

    - deriving definitions from experiences in constructing, visualizing, drawing, and measuring two-and three-dimensional figures, relating properties to figures, and contrasting and classifying figures according to their properties
    - identifying examples of

        * line, segment, ray, and angle (acute, right, obtuse, complementary, supplementary),
        * parallel, perpendicular, intersecting horizontal and vertical lines
        * base and altitude of polygons, radius, center, and diameter of circles
        * polygons and circles
        * important properties of polygons and circles, e.g., sum of angles in a triangle, opposite angles in a parallelogram are equal
        * three-dimensional figures and shapes, e.g., common polyhedrons

Observing and learning to represent two- and three-dimensional figures in various positions by drawing and construction helps students develop spatial sense.

2. *visualize and represent geometric figures in developing spatial sense*

   - using computer software to construct two-and three-dimensional shapes on a screen and then flip, turn, or slide them to view them from a new perspective
   - using properties of shapes, angles, and line relationships to draw models, both two-and three-dimensional
   - developing spatial sense by drawing and constructing two and three-dimensional figures.

3. *explore transformations of geometric figures*

   - using flips, slides, turns to study the concepts of congruence and similarity

4. *represent and solve problems using geometric model*

   - investigating two- and three- dimensional geometric models to understand the different growth rates for linear measures, areas, and volumes of similar figures
   - representing algebraic problems such as $(x + y)^2$ and the Pythagorean relationship by geometric models

A teacher's questioning techniques and language in directing students' thinking are critical to the students.

5. *understand and apply geometric properties and relationships*

   - using concepts of congruence, similarity, and transformation to compare lengths, area, and volumes of geometric shapes
   - using angle and line relationships to make comparisons and draw conclusions, both two- and three-dimensional
   - analyzing their thought processes and explanations, discussing the quality answers, and pondering such questions such as: Could it be another way? What would happen if...?
   - using correct vocabulary including such terms as: and, or, some, always, never, if...them, to reason, parallel, perpendicular, similar, etc.
   - using computer software to construct geometric figures and determine the measures of arcs, angles, and segments
   - using computer software to make conjectures and verify reasoning by exploring other figures

6. *develop an appreciation of geometry as a means of describing the physical world*

   - using symmetry in two- and three- dimensions to describe geometry in the world of art, nature, and construction.

STANDARD 13
Measurement

Measurement in grades 5-8 should be an active exploration of the real world. As students acquire the ability to use appropriate tools in measuring objects, they should extend these skills to new situations and new applications. The approximate nature of measure is an aspect of number that deserves repeated attention at this level. However, measurement activities in these grades should focus on using concepts and skills to solve problems and investigate other mathematical situations.

Measurement activities can and should require a dynamic interaction between students and their environment.

In grades 5-8, the mathematics curriculum should include extensive, concise experiences using measurement so that students can:

1. *extend their understanding of the process of measurement*

    - using examples from geometry to reinforce measurement concepts
    - using fractions, decimals, and rational numbers to represent measures
    - using the concept error of measurement

2. *estimate, make, and use measurements to describe and compare phenomena*

    - making and interpreting scale drawings
    - estimating measurements in both the metric and the customary system
    - using maps or scale drawings to approximate measurements and determine distances, directions, locations, areas, lengths, etc.
    - using many real-world measurements to generate data

3. *select appropriate units and tools to measure to the degree of accuracy required in a particular situation*

    - measuring in customary and metric units: length, capacity, mass or weight, and temperature
    - using conversion relations in measurement within the system being used, but not necessarily from one system to another

- using instruments of measurement to make drawings or identify an appropriate drawing to arrive at inductive conclusions
- selecting and using appropriate units from a variety of measurement systems to make defined measurements

4. *understand the structure and use of systems of measurement*

   - developing systems of measurement
   - understanding the difference between counting and measurement

5. *extend their understanding of the concepts of perimeter, area, volume, angle measure, capacity, and weight and mass*

   - developing procedures and using formulas to determine area and volume

   - applying concepts of area, perimeter or circumference, and volume, to appropriate measuring situations, given the formula

6. *develop the concepts of rates and other derived and indirect measurements*

   - using the concept of similarity to measure inaccessible objects
   - using the concept of rate in addition to speed in problem situations
   - using measurement of rate and explorations of rate relationships in problem situations

7. *develop formulas and procedures for determining measures to solve problems.*

Grades 9 - 12

INTRODUCTION

Due to variations in local regulations, the amount of mathematics a student will be exposed to in grades 9-12 will vary from district to district. Therefore, it becomes imperative that local districts have a common set of expectations upon which to base their curriculum decisions. The outcomes listed in this section should be included in every student's mathematical experience.

Certain assumptions form the basis for the standards and outcomes listed.

- The 9-12 curriculum topics are built on the foundation of the K-8 standards defined earlier. Students should be entering grade 9 with a common background and understanding of the K-8 skills and concepts.
- Whereas a certain level of computational proficiency will be expected of all students entering grade 9, investigation of the broad offering of topics should not be restricted for students who may be lacking this level of computational facility.

National Council of Teachers of Mathematics Standards . . . Grades 9 - 12
(with groupings indicated)

1. **Mathematics as Problem Solving**
2. **Mathematics as Communication (includes symbolism)**
3. **Mathematics as Reasoning**
4. **Mathematical Connections (includes modeling)**

5. **Algebra**
6. **Functions**

7. **Geometry from a Synthetic Perspective**
8. **Geometry from an Algebraic Perspective**

9. **Trigonometry**
10. **Statistics**
11. **Probability**
12. **Discrete Mathematical Topics**
13. **Conceptual Underpinnings of Calculus**
14. **Mathematical Structures/Foundations**

STANDARD 1
Mathematics as Problem Solving

Mathematical problem solving, in its broadest sense, is nearly synonymous with doing mathematics. It must permeate all of mathematics education and create the climate, or environment, within which mathematics is studied. Problem solving is more than applying specific techniques to certain types of word problems, more than an outline or series of steps for finding a solution. It is a process by which mathematics is studied, understood, and extended, and thus is the focus for school mathematics.

The need for problem recognition and solving is widespread. It can be found in questions that arise in everyday life, in other disciplines, and in business and industry. It is a part of mathematics, both in the motivation to study the subject itself, as well as a means to that study. The process of making wise decisions depends to a great extent on effectively solving problems. Technology can be used to perform routine calculations and iterative processes; human beings need to be able to think clearly, to reason logically, and apply the concepts of mathematics to real-world problem situations.

Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations and selecting appropriate strategies with which to find a solution. Problems become meaningful when they are set in a real-life context. Non-routine problems, in addition to the routine exercises traditionally found in textbooks, must also be included. Alternative solutions to selected problems provide an opportunity to look at the solution process and to consider other possible strategies. In many cases, there is no single correct answer. Open-ended problems requiring days, weeks, or some unspecified amount of time to solve are appropriate for the mathematics classroom. Students should also have the experience of problem posing–of recognizing and formulating their own problems. While many problems can be solved individually, small group work is appropriate and important to encourage cooperation, investigation, and communication.

Mathematical modeling is a situation in which mathematical problem solving is used in an abstract setting to simulate and model a real-world problem, thus providing clues and processes leading to a solution. Modeling, in this sense, is one of the highest forms of application of mathematics.

In grades 9-12, the mathematics curriculum should include the refinement and extension of methods of mathematical problem solving so that all students can:

1. *use, with increasing confidence, problem-solving approaches to investigate and understand mathematical content*

   - extending and connecting mathematical theory as a discipline

2. *apply integrated mathematical problem-solving strategies to solve problems from within and outside mathematics*

   - recognizing a problem when it exists and being able to select a skill or strategy to address the problem and arrive at an acceptable solution

3. *recognize and formulate problems from situations within and outside mathematics*

   - providing students with the tools and techniques to become mathematical problem solvers

4. *apply the process of mathematical modeling to real-world problem situations*

   - being able to make wise decisions as a result of problem-solving experiences
   - giving more than one possible solution to an open-ended problem situation.

STANDARD 2
Mathematics as Communication

All students in the 9-12 grades need to develop, learn, and use the language of mathematics. Students need to listen to, read about, write about, speak about, reflect upon, and demonstrate mathematical ideas. It is essential not to limit mathematical communication to exchanges between the teacher and the students. Students need extensive experience communicating with each other, both orally and in writing.

The role of the teacher needs to change from that of a dispenser of knowledge to that of a facilitator of students' learning through active participation. Small-group exploration of problem situations necessitates students communicating with each other. Teachers need to provide specific instruction concerning functioning in small groups, reading mathematics materials, and writing as a tool for facilitating learning. Students sometimes refuse to speak or write because they fear negative evaluations if they make errors. Teachers need to provide a climate that encourages students to take some risks and to experiment with non-traditional writing activities, such as keeping journals. Students need to realize that the first writing is not the final product, but that it can provide a useful recording of ideas that otherwise might be lost. Part of learning involves reflecting upon these ideas, rejecting some, using some, and modifying and developing others. Sharing and discussing ideas with other students can facilitate this process.

When students communicate their mathematical ideas, they recognize the advantages of using mathematical language and the importance of using mathematical terms and symbols correctly so that they can convey their ideas to others and understand what others are attempting to communicate to them.

Teachers may have to use specially designed activities and teaching strategies so that all students have the opportunity to develop their mathematical potential regardless of personal limitations, such as a lack of proficiency in the language of instruction.

In grades 9-12, the mathematics curriculum should include the continued development of language and symbolism to communicate mathematical ideas so that all students can:

1. *reflect and clarify their thinking about mathematical ideas and relationships*

2. *formulate mathematical definitions and express generalizations discovered through investigations*

    - with emphasis on the conceptual understanding of definitions and the ability to provide generalizations using the language of mathematics

3. *express mathematical ideas orally and in writing*

    - writing explanations of their work, not simply the answer, or even just "showing all the steps"
    - writing about the connections to other problems they have encountered
    - writing about specific difficulties encountered while solving problems
    - writing about their conjectures and proposals concerning extensions of their problems
    - posing questions clearly in written form

4. *read written presentations of mathematics with understanding*

5. *ask clarifying questions related to mathematics they have read or heard about*

6. *appreciate the economy, power, and elegance of mathematical notation and its role in the development of mathematical ideas.*

STANDARD 3
Mathematics as Reasoning

Reasoning is an integral part of mathematics. It is one of the fundamental attributes of mathematics. Reasoning connects mathematics to the real world, and it is at the root of problem solving. Reasoning should be included in all mathematics courses. All students should be able to determine examples and counterexamples, find patterns, and make and test conjectures. The potential for transfer between mathematical reasoning and the logic needed to resolve issues in everyday life can be enhanced by subjecting assertions about daily affairs to analysis. A concept that becomes increasingly important in grades nine through twelve is understanding that the connections among mathematical ideas facilitate students' ability to formulate and deductively verify conjectures across topics. This deductive reasoning can begin with two column proofs and quickly be converted to paragraph form. Reasoning also forms the basis for checking the reasonableness of an answer. The question of whether to overestimate or to underestimate is answered through reasoning.

All levels of instruction introduce and/or use simple valid arguments. Higher-order thinking skills are taught and evaluated in all courses. Different representations (graphs, tables of values, or equations) are used to draw conclusions about problem situations. The 9-12 program includes the basic tenets of logical argument and argument validation. Connections between mathematical ideas are used to develop arguments.

In grades 9-12, the mathematics curriculum should include numerous and varied experiences that reinforce and extend logical reasoning skills so that all students can:

1. *make and test conjectures*

    - designing a statistical experiment to study a problem and communicating the outcomes

2. *formulate counterexamples*

3. *follow logical arguments*

4. *judge the validity of arguments*

   - recognizing the limitations of conclusions or discriminatory data drawn from statistics, graphs, and/or tables
   - discovering fallacies in consumer advertising and propaganda, as well as data that might discriminate against gender, racial, or cultural differences
   - judging the validity of arguments

5. *construct simple valid arguments*

   - supporting a mathematical conclusion with logical reasoning through a written or oral justification
   - using logical reasoning techniques to draw conclusions, e.g., setting up an algorithm

6. *construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction*

   - exhibiting a working knowledge of the principles of logic, inductive and deductive reasoning, and the nature of proof
   - constructing formal proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction
   - proving elementary theorems with various mathematical structures, such as groups and fields.

STANDARD 4
Connections

The connections among mathematical topics and between mathematics and other disciplines are referenced in several standards. Two general types of connections are important: (1) modeling the connections between possible problems that may arise in the real world or in other disciplines and their mathematical representation(s); and (2) mathematical connections among equivalent representations and among corresponding processes. Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept have a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics.

Students should constantly be encouraged to look back at the solution process to consider other possible strategies. Such an approach helps establish mathematical connections as well by focusing students' attention on commonalities across different mathematical representations. As a means of emphasizing these connections between mathematical ideas, new concepts should be introduced, where possible, as extensions of familiar mathematics.

The connections between algebra and geometry are among the most important in high school mathematics. Computer and graphing-calculator technology now make it possible to exploit these connections; many problems traditionally solved using algebra can now be solved efficiently and using the geometric representation and computer/graphing techniques. This approach reduces the need for algebraic manipulative skills as a prerequisite, thereby allowing all students to address interesting problems and explore important mathematical ideas.

In grades 9 - 12, the mathematics curriculum should include investigation of the connections and the interplay among various mathematical topics and their applications so that all students can:

1. *recognize equivalent representations of the same concept*

   - recognizing similarities and differences among mathematical strategies, systems, and concepts

2. *relate procedures in one representation to procedures in an equivalent representation*

   - seeing new concepts as extensions of more familiar mathematical topics

3. *use and value the connections among mathematical topics*

   - focusing on multiple methods of solution for a given problem and commonalities across different mathematical representations

4. *use and value the connections between mathematics and other disciplines*

   - being able to apply mathematical formulas in the areas of investment and consumer-related topics.

STANDARDS 5-6
Algebra and Functions

Algebra is the language through which most of mathematics is communicated. It also provides a means of operating with concepts at an abstract level and then applying them, a process that often fosters generalizations and insights beyond the original context.

Topics and outcomes in this area are designed to expand and enhance the algebraic concepts included in Standards 8 and 9 for grades 5-8. The understanding of algebraic representation is a prerequisite for more formal work in mathematics as well as a useful tool for applying mathematics in other curriculum areas. Functions serve as a unifying concept across all mathematics courses.

The emphasis in grades 5-8 is on the exploration of functions and patterns and on using algebraic manipulations to represent and solve problems. In grades 9-12, the emphasis on the conceptual understandings of functions and algebra includes a means of representing real-world situations; meanings of linearity, equivalence and solutions; and the informal development of concepts such as limit, maximum, and minimum. For example, students should spend less time plotting curves point by point and more time interpreting graphs, exploring the properties of graphs, and comparing the relationships that exist between graphs and equations.

The use of appropriate technology allows for and supports a richer set of algebraic experiences and investigations for all students. Scientific calculators replace the need for skills in reading and interpolating tables. Computer software and graphing calculators can replace the tedium of pencil-and-paper graphing and aid in analyzing the resulting systems. Comparisons and modifications of graphs resulting from changes in parameters and values of the constant provide a basis for more meaningful generalizations.

In grades 9-12, the mathematics curriculum should include the continued study of algebraic concepts and functions so that all students can:

1. *model real-world phenomena with a variety of functions*

   - investigating a variety of functions, such as circular and other periodic functions
   - representing problem situations with geometric models and applying properties of two- and three-dimensional figures
   - recognizing and formulating problems within and outside of mathematics
   - applying the process of mathematical modeling to real-world problem situations

2. *represent situations that involve variable quantities with expressions, equations, and inequalities*

   - recognizing formulas, equations, variables, constants, expressions, exponents, and inequality
   - comparing two ratios and generating sets of equivalent ratios, using numbers and/or variables

3. *operate on expressions and matrices, and solve equations and inequalities*

   - solving linear, quadratic, and polynomial equations
   - operating on expressions and solving polynomial equations and inequalities
   - solving systems of equation, both linear and quadratic
   - determining, given a value for two variables, the third variable in a formula, such as $a = bc$ or $a = b+c$
   - solving for the missing value, given any three values in proportion, using numbers and/or variables

4. *translate among tabular, symbolic, and graphical representations of functions, and analyze and interpret relationships using tables, rules, and graphs*

   - interpreting and drawing spatial phenomena
   - drawing and constructing accurate figures and scale drawings
   - recognizing symmetric relationships found in graphs

5. *recognize that a variety of problem situations can be modeled by the same type of function*

6. *analyze the effects of parameter changes on the graphs of functions*

and so that, in addition, college-intending students can:

7. *understand operations on, and the general properties and behavior of, classes of functions*

8. *use matrices to solve linear systems*

9. *demonstrate technical facility with algebraic transformations, including techniques based on the theory of equations.*

STANDARDS 7-8
Geometry

Geometry should provide opportunities, both synthetically and algebraically, to deepen students' understanding of shapes and their properties. The wide applicability of geometry in human life activities should be emphasized. It is assumed that students have experienced repeated examples of geometry in recreations, practical construction tasks, in the sciences, and in the arts.

High school geometry needs to build on the conceptual foundation started in the earlier grades. Two- and three-dimensional models should be used to help students visualize and understand spatial relationships that are fundamental to everyday life and many careers. Inductive reasoning can be used to help students develop geometrical intuition for work with more abstract ideas. The interplay between geometry and algebra will strengthen students' abilities to formulate and analyze problems both within and outside mathematics. Geometry provides students with the opportunity to learn the basic tenets of logical argument and to validate arguments. Development of these arguments will be aided by using various mathematical representations, stressing the algebraic and geometric connections. Students need to understand that specific problems are often solved more easily by selecting another system.

In grades 9 - 12, the mathematics curriculum should include the study of the geometry of two and three dimensions from both an algebraic and a synthetic point of view so that all students can:

1. *interpret and draw two- and three-dimensional objects to represent problem situations with geometric models and apply properties of figures*

    - identifying examples of
        - line, segment, ray, and angle (acute, right, obtuse, complementary, supplementary)
        - parallel, perpendicular, intersecting horizontal and vertical lines
        - base and altitude of triangles, radius, center, and diameter of circles
        - polygons and circles
        - important properties of polygons and circles, e.g., sum of angles in a triangle, opposite angles in a parallelogram are equal
        - three-dimensional figures and shapes, e.g., common polyhedrons

- applying concepts of area, perimeter, or circumference, and volume to appropriate measuring situations, given the formulas
- using angle and line relationships to make comparisons and draw conclusions, both two- and three- dimensional
- using properties of shapes, angles, and line relationships to draw models, both two- and three-dimensional
- using instruments of measurement to make drawings or identify an appropriate drawing to arrive at conclusions

2. *classify figures in terms of congruence and similarity and apply these relationships with and without transformations, with and without coordinates*

   - recognizing the various types of symmetries of a figure
   - using concepts of congruence, similarity, and transformations to compare lengths, area, and volumes of geometric shapes

3. *translate between synthetic and coordinate representations*

   - representing algebraic relationships as objects and relations in geometry, using ordered pairs, equations of lines, etc.
   - representing geometric relationships in algebraic terms

4. *analyze properties of Euclidean transformations and relate translations to vectors*

and so that, in addition, college-intending students can:

5. *develop an understanding of an axiomatic system through investigating and comparing various geometries*

6. *deduce properties of figures using vectors*

7. *apply transformations, coordinates, and vectors in problem solving.*

STANDARD 9
Trigonometry

Trigonometry should underscore the connnections between geometry and algebra. All students should apply trigonometric methods to practical situations involving triangles. Trigonometry not only remains an important and powerful tool for science and engineering but also continues to provide an aesthetic attraction for many students through its regularities and symmetries. The use of technology makes both aspects of the subject readily accessible to most students. Scientific and graphing calculators can improve the teaching of trigonometry by providing more class time, computational power to develop conceptual understandings, and realistic applications. This in turn provides opportunities for greater integration of trigonometry with geometry and algebra.

Four general areas to be integrated in the teaching of trigonometry include:

(i) problems chosen in such a way as to integrate various strands of mathematics with applications from other curricular areas
(ii) emphasis on the use of calculators rather than table-reading skills and interpolation
(iii) development of trigonometric function concepts progressing from the informal to the formal
(iv) data from the real-world situations to illustrate the properties of trigonometric functions.

In grades 9-12, the mathematics curriculum should include the study of trigonometry so that all students can:

1. *apply trigonometry to problem situations involving triangles*
   - recognizing and formulating problems from within and outside mathematics
   - applying the process of mathematical modeling to real-world problem situations
2. *explore periodic real-world phenomena using the sine and cosine functions*

and so that, in addition, college-intending students can:

3. *understand the connection between trigonometric and circular functions*
4. *use circular functions to model real-world phenomena*

5. *apply general graphing techniques to trigonometric identities*

   - translating among tabular, symbolic, and graphic representations of functions
   - analyzing the effects of parameter changes on the graphs of functions

6. *understand the connections between trigonometric functions and polar coordinates, complex numbers, and series.*

STANDARD 10
Statistics

Collecting, representing, and processing data are activities of major importance in contemporary society. Summarizing, analyzing, and transforming data are statistical techniques that link mathematics to other subject areas. Students need to apply the skills of sampling, curve fitting, hypothesis testing, and drawing inferences in solving problems and evaluating the statistical claims they encounter in their daily lives.

The study of statistics in grades 9-12 consolidates, deepens, and builds on the understanding of exploratory data analysis as developed in the elementary and middle grades. Application of these statistical tools both within the context of the school setting and in out-of-school activities provides meaningful experiences as well as exploratory studies which can be immediately useful. Statistical data, summaries, and inferences appear more frequently in the work and everyday lives of people than any other form of mathematical analysis. Therefore, it is essential that statistics be given a more prominent position in the high school curriculum.

Mathematical thinking becomes a critical tool in looking at appropriate and inappropriate use of statistics. The opportunity to analyze the validity of statistical conclusions and the use, misuse, and abuse of data in the context of societal applications becomes important in developing literate citizens. The study of statistics should provide students with an understanding of which measures are appropriate for a given problem and what such measures as mean, variance, and correlation can tell them about a problem.

Computing technology provides accurate representations of graphs and rapid and precise calculations of statistical measures allowing time to develop a better understanding of their meaning and an interpretation of the results.

In grades 9-12, the mathematics curriculum should include the continued study of data analysis and statistics so that all students can:

1. *construct and draw inferences from charts, tables, and graphs that summarize data from real-world situations*

    - recognizing and formulating problems within and outside of mathematics
    - constructing a variety of representations of a given set of data, such as tables, charts, graphs, box plots, etc.

2. *use curve-fitting to predict from data*

   - using a scatter plot to arrive at a curve or line that best fits the data

3. *understand and apply measures of central tendency, variability, and correlation*

   - computing measures of central tendency and dispersion, e.g., mean, median, mode, range

4. *understand sampling and recognize its role in statistical claims*

   - selecting an appropriate sampling method for given statistical analysis

5. *design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes*

   - completing a relevant task when given appropriate directions or questions to address, for example:

     * determining a statement or theory to be tested
     * deciding what data must be obtained
     * differentiating between biased and unbiased data
     * devising a plan to obtain and record the data
     * organizing the data in presentable form
     * using the data to draw conclusions
     * explaining and justifying the conclusions
     * reevaluating the conclusions when new data becomes available

6. *analyze the effects of data transformations on measures of central tendency and variability*

and so that, in addition, college-intending students can:

7. *transform data to aid in data interpretation and prediction*

8. *test hypotheses using appropriate statistics.*

STANDARD 11
Probability

Probability provides concepts and methods for dealing with uncertainty and for interpreting predictions based on uncertainty. Students should extend their K-8 experiences with simulations and experimental probability to continue to improve their intuition. These experiences provide students with a basis of understanding from which to make informed observations about the likelihood of events, to interpret and judge the validity of statistical claims in view of the underlying probabilistic assumptions, and to build formal concepts of theoretical probability.

Differences between experimental and theoretical probability need to be explored along with an intuitive approach to independent and dependent events and their relationship to compound events and conditional probability. Students should also understand the difference between, and the advantages associated with, theoretical and simulation techniques. Experiences drawn from the real world and situations meaningful to the student are important in an instructional context.

In grades 9-12, the mathematics curriculum should include the continued study of probability so that all students can:

1. *use experimental or theoretical probability, as appropriate, to represent and solve problems involving uncertainty*

    - solving enumeration, counting, and finite probability problems
    - identifying common use of the terms: odds, chance, random, probability

2. *use simulations to estimate probabilities*

3. *understand the concept of a random variable*

4. *create and interpret discrete probability distributions*

    - constructing a sample space and determining the probability (chance) of an event happening, e.g., the occurrence of a head when flipping a coin, the occurrence of drawing a certain object from a given set

5. *describe, in general terms, the normal curve and use its properties to answer questions about sets of data that are assumed to be normally distributed*

and so that, in addition, college-intending students can:

6. *apply the concept of random variable to generate and interpret probability distributions including binomial, uniform, normal, and chi-square.*

STANDARD 12
Discrete Mathematics

Discrete mathematics is the study of mathematical properties of sets and systems that have only a finite number of elements. The physical world is often modeled by continuous mathematics. Therefore, the school curriculum has emphasized algebra, geometry, trigonometry, and the prerequisite ideas for the study of calculus. Because digital computers are essentially finite, discrete machines, the presence of increasing computer technology increases the need for an emphasis on discrete mathematics topics such as finite sequences, recurrence relations, and algorithm design.

Courses entitled "discrete mathematics" and "finite mathematics" have traditionally been taught at the college level. However, much of the content of these courses is appropriate for grades 9-12 and provides a rich source of examples involving pattern searching and problems which involve the use of non-continuous functions, such as the greatest integer part. Frequently, courses in discrete mathematics and finite mathematics include topics associated with the study of probability, such as enumeration techniques and permutations and combinations. The study of discrete mathematics topics can be integrated into existing mathematics courses, integrated into new courses as mathematics curricula evolve, and/or used to define a discrete mathematics course that is appropriate for all secondary students.

In grades 9-12, the mathematics curriculum should include topics from discrete mathematics so that all students can:

1. *represent finite sets with appropriate notation and use basic set relations and operations, including 1:1 correspondence, equivalence, subset, union, intersection, and complement*

    - applying DeMorgan's Laws and the distributive properties of union over intersection and intersection over union

2. *represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations*

    - determining when to apply non-continuous functions to appropriate problem-solving situations, for example: "How many costumes can be made from x yards of material if each costume requires three yards?" or "How many gallon cans of paint should be purchased to paint a wall with an area of x square meters if the contents of a one gallon can cover 12 square meters?"

- using finite graphs to represent problem situations, e.g., communications networks and pipeline flow problems
- applying Eulerian circuits to problems, e.g., the Konigsberg Bridge problem and the mail carrier's route problem
- applying Hamiltonian circuits to problem situations, e.g., the traveling salesperson problem
- applying incompatibility graphs to problems, e.g., scheduling meetings of committees of an organization in which some of the members might be assigned to more than one committee
- representing finite graphs using matrices
- developing and using recurrence formulas to study compound interest problems and to generate terms for various types of sequences, e.g., arithmetic, geometric, Fibonacci

3. *solve enumeration and finite probability problems*

   - applying the pigeonhole principle to appropriate problem situations, for example, "At a camp, if each of ten campers are allowed to select one of three colored t-shirts and one of three colored shorts, show that at least two of the campers must have identically colored t-shirt and shorts outfits."
   - determining the number of permutations of a finite number of elements

and so that, in addition, college-intending students can:

4. *represent and solve problems using linear programming and difference equations*

5. *apply truth tables to problems such as simple combinatorial circuits*

6. *apply mathematical induction to appropriate problems*

7. *solve a problem by creating, coding, and validating a computer algorithm*

STANDARD 13
Underpinnings of Calculus

This standard does NOT advocate the formal study of calculus in high school for all students or even for many college-intending students. However, the physical world is most often modeled by continuous mathematics, that is, the calculus and its prerequisite ideas. The presence of calculators and computers provides opportunities for students to develop a conceptual understanding of the key ideas and applications of some of the types of problems for which calculus was developed. Thus it is important to stress that this standard is addressing the foundations of calculus and not specific skills such as differentiation and integration covered in calculus courses.

In grades 9-12, the mathematics curriculum should include the informal exploration of calculus concepts from both a graphical and a numerical perspective so that all students can:

1. *determine maximum and minimum points of a graph and interpret the results in problem situations*

   - generating graphs using graphing calculators or computer software, and reading the maximum and minimum points on the graph

2. *investigate limiting processes by examining infinite sequences and series and areas under curves*

   - using calculators to explore infinite sequences and determining if they will converge and if so, finding the limit
   - approximating the area under a finite portion of a continuous curve by partitioning the region into appropriate rectangles and summing the areas of the rectangles, and finding upper and lower bounds for the area

and so that, in addition, college-intending students can:

3. *understand the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line, and their applications in other disciplines*

   - informally analyzing graphs, e.g., discussing the slope of the tangent line to a curve, comparing the slopes of f(x) and (f(x) + a) for particular values of x, and informally exploring and interpreting the steepness of the slope and the concavity in a graph of distance plotted against time

- using graphing calulators and/or computing software to enlarge a small portion of a graph so that it appears to be a straight line and determining the slope of that portion of the graph
- approximating the area under the finite portion of a continuous curve by means other than the rectangular approximation and discussing advantages and disadvantages of trapezoidal methods

4. *analyze the graphs of polynomial, rational, radical, and transcendental functions*

   - recognizing that polynomial functions, P(x), will be smooth continuous curves without any cusps, discussing what happens to the graph of a polynomial function as the independent variable becomes infinitely large
   - determining the y-intercept and, if the polynomial is in factored form, determine the x-intercepts
   - predicting effect on the appearance of a graph of replacing x with x + a or adding a constant.
   - discussing the symmetries of graphs
   - generalizing the comparison of the graphs of $y = x^m$ and $y = x^n$ if m and n are integers and m>n and the relation between the graphs of $y = x^n$ and $y = x^{(\frac{1}{n})}$ when n is a positive integer
   - understanding that a polynominal function is a special type of a rational function and predicting when rational functions will have vertical asymptotes
   - understanding the difference between functions such as $y = (x^2 - 4) / (x-2)$ and $(x^2 - 3) / (x - 2)$
   - comparing the graphs of rational functions in which the degree of the numerator is less than, equal to, and greater than the degree of the denominator
   - becoming familiar with the properties of basic trigonometric functions, exponential functions, and logarithmic functions

STANDARD 14
Mathematical Structures/Foundations

The structure of mathematics is like the steel framework of a modern building. Students should become aware of this structure, how it provides a strong foundation on which a variety of content strands are built, and how it simultaneously holds these different strands together. For example, one of the girders in this building is the associative property to which are attached objects and operations in such wide-ranging mathematical subjects as arithmetic, algebra, functions, and geometric transformations. An awareness of these broad structuring principles frees students to take a more constructive approach to new mathematical topics and provides them with a framework involving ideas or concepts that facilitates long-term retention.

Mathematical structure in the form of lists of general properties is *not* a good starting point for instruction. Rather, students gain a sense of the structure of mathematics over an extended time period through the general accumulation of experience, as well as through more focused activities. It is neither necessary nor appropriate for them to hear constantly the word *"structure"* applied to their activities; occasional summary statements will serve them far better. It is also essential to recognize that mathematical structure and formalism are not synonymous. In mathematics, just as with a building, all students can develop an understanding and appreciation of its underlying structure independent of a knowledge of the corresponding technical vocabulary and symbolism. The degree of formalism must be consistent with the student's level of mathematical maturity.

In grades 9 - 12, the mathematics curriculum should include the study of mathematical structure so that all students can:

1. *compare and contrast the real number system and its various subsystems with regard to their structural characteristics*

2. *understand the logic of algebraic procedures*

3. *appreciate that seemingly different mathematical systems may be essentially the same*

    - recognizing applications of matrices as representation tools leading naturally to the usual definitions of equality and addition
    - considering the set of 2x2 matrices with integer entries under addition and exploring whether the commutative, associative, identity, and inverse properties hold

and so that, in addition, college-intending students can:

4. *develop the complex number system and demonstrate facility with its operations*

5. *prove elementary theorems within the various mathematical structures, such as groups and fields*

    - recognizing that the existence of a unique inverse for a given 2x2 matrix is equivalent to the existence of a unique solution of a related system of two linear equations in two variables

6. *develop an understanding of the nature and purpose of axiomatic systems*

    - comparing the structure of 2x2 matrices with integral entries under addition with other systems such as the integers under addition
    - exploring 2x2 real matrices using the usual definitions of matrix addition and multiplication as an example of a system in which multiplication is not commutative.

Appendices

APPENDIX A

COMPONENTS OF MATHEMATICS REFORM

Six identified issues emerged during the research and development phase of the learner outcome process. The first four appeared in Chapter 2 of the 1988 *Model Learner Outcomes for Mathematics Education* document. In the revision process, two additional issues seeméd to need emphasis as well. The issues are identified and listed on the last page of Chapter 2 of this document. Further development and expansion of these critical issues is included for clarification and discussion purposes.

ROLES OF TECHNOLOGY

The age of information and technology is here. Business, education, and industry jostle each other to be the leader in how technology should best be used. Maybe at a later date a clear picture will emerge of the role of education in forming a technologically literate person. Without a clear picture, education has vacillated between teaching how to turn on a computer, teaching obsolete programming languages, and using a computer as a tool to learning.

Within the process of formulating the learner outcomes in mathematics, at least two definite strands emerged: there must be an increase in the appropriate use of computers and calculators; and they must be used as a means of discovery and learning, not an end. In other words, use more technology but do not teach technology. The mathematics classroom is not the place for computer facts and jargon development, but rather the mathematics classroom must use the power inherent in the technology to teach mathematics in a more efficient, equitable, thoughtful, and effective manner. Extra effort may have to be expended to ensure equity between sexes and among roles of cultural backgrounds.

It should be noted that proclaiming more use of technology does not automatically result in more use for a variety of legitimate reasons. One immediate reason for hesitancy is that teachers often do not know how to use technology. They need to critically examine how they would intelligently use the present technology available in schools. Our immediate past has some glaring failings in jumping on a bandwagon and later regretting the meager knowledge we had in how to implement the change on a daily basis. Caution should be used in suggesting technology as a panacea for any ill or problem infecting society. More research is needed regarding technology's application and current teachers need further training in its use.

As with higher order thinking skills, the role of technology impacts both the elementary and secondary schools. The implications for each level may be somewhat different. A few of the implications for each level will be stated to explain the impact on the curriculum.

Implications for elementary schools: Teachers will—

1. Learn about and implement both the calculator and computer at all grade levels. The arguments about what needs to be learned before touching a calculator implies an inappropriate use of the calculator.

2. Discard many paper-pencil algorithms as unnecessary, e.g., long division; expanded multiplication.

3. Use more real-world number crunching applications of arithmetic at earlier grades with calculators, such as data collection and analysis.

4. Use the graphics capability of various forms of technology to enhance learning of certain concepts.

5. Emphasize the "why and when" of arithmetic skills more than the "how."

6. Spend more time teaching guessing, estimating, and reasonableness of results.

7. Decrease individual seatwork and increase small group projects, exploring interesting activities with the use of technologies, e.g., create a game with unfair rules and change the rules to be fair; estimate the cost of a group meal.

Implications for secondary schools: Teachers will—

1. Allow every learner to have equal access to a scientific calculator from seventh to twelfth grade.

2. Spend more time in all mathematics classes conjecturing, guessing, and exploring through the use of available technology.

3. Use the graphics capability of various forms of technology to explore interesting situations with functions, geometry, and analysis.

4. Encourage software companies to develop quality utilities and necessary support materials for classroom applications that are multicultural and gender fair.

5. Allow or encourage the use of calculators in testing situations. Forbidding the use of calculators in test settings gives the learner the wrong impression that technology is not a help but a hindrance.

6. Consider what role programmable calculators will play in advanced courses such as statistics, analysis, and calculus. There may be a need to revise the course content to more fully use the available technology.

7. Investigate the need to introduce more discrete mathematics in the secondary mathematics curriculum. In many situations, discrete mathematics draws upon present-day technology to highlight and expand the understanding of the concepts being presented. Therefore, all learners in secondary school must be adequately prepared in both discrete mathematics and the use of present-day technology.

TEACHERS WITH ADEQUATE MATHEMATICS PREPARATION

During the last 30 years, tremendous progress has been made in the area of teacher preparation. Present-day teachers of secondary mathematics have had more mathematics courses and teaching methods in mathematics than ever before in teacher education history. In an international comparison study (Second International Mathematics Study 1982), United States mathematics teachers at both the eighth- and twelfth-grade levels either exceeded or compared favorably with the other countries studied in terms of mathematics content and pedagogy courses taken in college. Therefore, it would seem that United States' mathematics teachers at the secondary level are for the most part adequately prepared in most traditional areas of mathematics teaching. However, the same statement cannot be made for elementary teachers.

Presently, many elementary teachers lack the confidence, background knowledge, and preparation in mathematics to be effective teachers of mathematics. Progress in students' performances will not occur until all elementary teachers are better prepared in mathematics and develop equal expectations for all students despite cultural and gender differences. This must include ongoing inservice as well as preservice training. National recommendations suggest teachers be specifically prepared in mathematics in grades four and beyond. In particular, the mathematics specialist would teach all the mathematics beginning no later than grade four and also supervise the instruction in mathematics at the earlier grades.

The recommendation to have adequately prepared elementary school teachers in mathematics is not a new recommendation, but the need for change is imperative and should be seriously discussed now within the district level as well as at the state level of policymaking. Local school districts must encourage their elementary teachers to continue their professional development in mathematics, particularly to prepare them to assume a leadership role in curriculum revision in the district. A mentoring program for preservice and new teachers, with more careful supervision, could also strengthen mathematics instruction.

Being a teacher of mathematics at any level demands a certain sophistication in the content being taught. The mathematics curriculum at the upper elementary level is complex, sophisticated, and diverse. Teachers with special preparation for this level of instruction would enhance the learning of mathematics. In addition, these specially prepared teachers would also spend part of the day or week on staff development, curriculum planning, and assisting lower-grade teachers in the area of mathematics instruction.

Teacher preparation in mathematics is closely linked to a successful program in mathematics for all levels of education. This is particularly true for the elementary school--more mathematics content preparation should be encouraged for elementary school teachers.

HIGHER ORDER THINKING SKILLS

The phrase "higher order thinking skills" is in vogue at the present time. Any educator who wants to be listened to and respected would be foolish to argue against teaching higher order thinking. In 1985, the Minnesota Department of Education and the Minnesota Council of Teachers of Mathematics held a conference to draft a position paper *Higher Order Thinking Skills and Mathematics Education (1985).* The following excerpt is taken from that paper:

> Thinking skills are the dynamic mental processes, both intuitive and logical, used in the collecting, organizing, interpreting, and applying information for the purpose of arriving at decisions and/or gaining new knowledge.

Clearly, mathematics is not the only discipline taught in the schools which would claim thinking skills as within its purview, but mathematics does provide a fertile environment for exploring and employing thinking skills.

From common sense, one can conclude that thinking itself is a basic skill, one that is certainly not obsolete. Thinking is essential for meaningful human endeavor and fullness of living. It is necessary for success in living, for future careers, and for retraining in different careers. Clear, logical thought, and the effective expression of those thoughts, are highly valued and sorely needed in our complex and diverse society. Mathematics should be taught as a way of thinking rather than only a collection of isolated bits of information or rules. Increasingly, mathematics is a language of communication in society.

The arithmetic mentality (computation-only orientation) seems to blind many people when they discuss mathematics. Their view is clouded by our own failure to show that mathematics is a way of thinking, a model for organizing thought, and a discipline with powerful logical thinking tools.

An emphasis must be placed on viewing mathematics holistically over the entire educational career of a learner and not serially from class to class or course to course. One goal of mathematics education in the state of Minnesota must be that graduates have consistently been exposed to a board view of mathematics. Presently, the view by many graduates is narrow and focused on rules. The mathematics curriculum must be revised to show the discipline of mathematics as an entity rather than segments. An integrated view of mathematics rather than separation into topics such as algebra and geometry would expand the perspective of learners. The thread of thinking skills throughout the curriculum may be the glue which holds the integrated mathematics curriculum together.

The major thrust toward encouraging higher order thinking skills impacts both the elementary and secondary schools. The implication for each level may be somewhat different. A few of the implications for each level will be stated to illustrate the impact on the curriculum.

Implications for elementary schools. Teachers will—

1. De-emphasize paper and pencil algorithms such as long division and multidigit multiplication since calculators have eliminated the need for the skill (though not the concepts).

2. Re-allocate time to emphasize developing concepts rather than practicing skills.

3. Spend more class time in understanding mathematics rather than practicing algorithms.

4. Use writing in mathematics as a way to clarify thinking and communicate ideas. Include opportunities to write during the mathematics class or to write about mathematics during creative writing times.

5. Sharpen and refine thinking skills in small group settings. Organized, structured, and flexible small groups, cross cultural and cross gender in their makeup, should be commonplace during the mathematics class time. Team effort and team thinking is basic to success in most careers.

6. Include mental computation with oral responses in daily activities.

7. De-emphasize speed in mathematics and emphasize meaningful reasons why certain patterns and structures emerge in certain situations.

Implication for secondary schools: Teachers will—

1. Focus on informal logic and deductive thinking on a regular basis (not one unit) in junior high mathematics classes. These learners are struggling to be formal reasoners and are searching for ways to structure their thought processes.

2. Emphasize understanding and mastery while decreasing dramatically review material in seventh- and eighth-grade mathematics, replacing it with more relevant content, e.g., problem solving, number theory, geometry, statistics.

3. Overhaul the entire curriculum to show a continuous curriculum for a variety of diverse learners with multiple career paths, while addressing the cultural and gender differences.

4. Show the interrelatedness of the branches of mathematics, both from a content and cultural context.

5. Make mathematics and deductive reasoning a way of thinking throughout the curriculum, not only in a geometry course.

6. Develop proofs in other ways than two columns—develop a flow pattern proof, a paragraph proof.

7. Ensure that learners take mathematics to the limit of their ability during every year in school, making the curriculum relevant for every learner and impressing upon each the power of clear thinking. In particular, females and minorities, for whom expectations have in many cases been different, need to be supported on their achievement and encouraged to persevere in the study of mathematics.

COGNITIVE DEVELOPMENT

Over the last two decades, much has been written about the intellectual development of the learner from birth to at least 18 years old. However, education seems to be slow in revising and updating the curriculum to reflect what we know about learner cognitive development. There is a concern that perhaps one reason children do not learn mathematics well is that it is presented at a rapid pace with abstract, symbolic concepts, and little recognition of levels of individual cognitive development. Presentation should rather match each learner's stage of development. This requires a constant variety of instructional techniques to accommodate learners of all types and all stages. Special consideration should be given to the needs of those traditionally excluded from learning mathematics such as many females and minority students.

The model learner outcomes in Chapter III assume that the curriculum will be developed to assure a correct placement of topics and instructional strategies as well as accessibility and equity. Without this, many good intentions for curriculum revision and reform will be in vain.

Cognitive and content development aspects should be considered together in any curriculum project. It should be evident that one's cognitive level determines what one is able to learn as well as what constitutes appropriately challenging instruction for that person.

APPENDIX B
Mathematical Association of America and National Council of Teachers of Mathematics Position Report On Calculus in High School

A single variable calculus course is now well established in the 12th grade at many secondary schools, and the number of students enrolling is increasing substantially each year. In this letter, we would like to discuss two problems that have emerged.

The first problem concerns the relationship between the calculus course offered in high school and the succeeding calculus courses in college. The Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) recommend that the calculus course offered in the twelfth grade should be treated as a college-level course. The expectation should be that a substantial majority of the students taking the course will master the material and will not then repeat the subject upon entrance to college. Too many students now view their twelfth grade calculus course as an introduction to calculus with the expectation of repeating the material in college. This causes an undesirable attitude on the part of the student both in secondary school and in college. In secondary school all too often a student may feel "I don't have to master this material now, because I can repeat it later;" and in college, "I don't have to study this subject too seriously, because I have already seen most of the ideas." Such students typically have considerable difficulty later on as they proceed further into the subject matter.

MAA and NCTM recommend that all students taking calculus in secondary school who are performing satisfactorily in the course should expect to place out of the comparable college calculus course. Therefore, to verify appropriate placement upon entrance to college, students should either take one of the Advanced Placement (AP) Calculus Examinations of the College Board, or take a locally administered college placement examination in calculus. Satisfactory performance on an AP examination carries with it college credit at most universities.

The second problem concerns preparation for the calculus course. MAA and NCTM recommend that students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry. This means that students should have at least four full years of mathematical preparation beginning with the first course in algebra. The advanced topics in algebra, trigonometry, analytic geometry, complex numbers, and elementary functions studied in depth during the fourth year of preparation are critically important for students' latter courses in mathematics.

It is important to note that at present many well-prepared students take calculus in the twelfth grade, place out of the comparable course in college, and do well in succeeding college courses. Currently, the two most common methods for preparing students for a college-level calculus course in the twelfth grade are to begin the first algebra course in the eighth grade or to require students to take second year algebra and geometry concurrently. Students beginning with algebra in the ninth grade, who take only one mathematics course each year in secondary school, should not expect to take calculus in the twelfth grade. Instead, they should use the twelfth grade to prepare themselves fully for calculus as freshmen in college.

We offer these recommendations in an attempt to strengthen the calculus program in secondary schools. They are not meant to discourage the teaching of college-level calculus in the twefth grade to strongly prepared students.

Lynn Arthur Steen
President
Mathematical Association of America

John A. Dossey
President
National Council of Teachers of Mathematics

APPENDIX C

Mathematical Sciences in Minnesota

Executive Summary

Mathematical Sciences in Minnesota is a statewide study of the role of the mathematical sciences in Minnesota—perhaps the first of its kind in the nation. This study examines a wide variety of issues, including:

- Curricular articulation among the various educational levels;
- Characteristics of the market for persons with mathematical training;
- Supply, demand, and features of teacher preparation programs;
- Applications of mathematical research in Minnesota industry;
- Impact on Minnesota of new national standards for mathematics education.

The study is intended to help inform policy recommendations to agencies in the State of Minnesota, as well as to provide a model for other states of how best to determine priorities and needs in the mathematical sciences.

The scope of this study is broad, ranging from mathematics education to mathematical research and from primary school to graduate school. Mathematics education forms a single but highly complex system spanning all educational levels from the earliest grades to adult education. In the world of mathematics education, life-long learning is not just a slogan but a reality.

As a first venture of its kind, the papers in this volume do not claim to provide a definitive analysis of the many interrelated issues. Authors very often found that important data was unavailable, or inconsistent. Different writers approached issues from different perspectives, sometimes leading to conflicting conclusions. More often than not, the papers raise more issues than they settle. They do, however, highlight important information that must be considered by those who set policy for mathematics education in the state. Observations in the papers provide a framework for various recommendations, which take on increased urgency when one examines the future needs of Minnesota in terms of present demographic and educational predictors. In what follows we summarize many of the findings, conclusions, and recommendations found in the papers of this study, frequently in the actual language of the reports themselves.

Every educational and developmental state is a potential point of intervention in the educational pipeline for mathematics, science, and engineering. Other states are tackling the issue of making mathematics and science education more hospitable to larger numbers of individuals—especially to women and minorities—with a zeal that may not yet be recognized in Minnesota. It is time to marshall support for a mobilization of Minnesota mathematics.

Mathematical Sciences in Minnesota—A State Assessment, Minnesota Mathematics Mobilization, 1990. Permission Granted.

Priorities for the 1990s

Goals

- To ensure that all graduates of Minnesota high schools are mathematically literate as described in the NCTM *Standards*.
- To raise the mathematical preparation of Minnesota elementary and middle school teachers to match MAA-NCTM guidelines.
- To raise the percentages of women, minorities, and the handicapped in mathematics at each educational level to their percentages in the State's population.
- To align state and local assessment with the broad objectives of mathematics education.

Principles

- Actions to achieve equity and excellence in K-12 mathematics must begin with the expectation that all students can and will learn mathematics.
- All students should be taught by teachers who like mathematics, who know it well, and who know how to teach it.
- Learning mathematics should be an active, constructive process based on realistic problems and making full use of computers and calculators.
- All teachers must have regular opportunities for learning about changes in mathematics, about new applications, and about effective methods of teaching.

Recommendations

- Introduce a mathematics specialist license for elementary schools and require that schools employ specialists.
- Bring standards for teacher certification into compliance with the MAA-NCTM Standards for the Mathematical Preparation of Teachers.
- Increase educational investment in non-college bound students to assure that they are well educated and technically prepared for future demands of the workforce.
- Support a sufficient number of summer institutes to provide an appropriate mathematical enrichment experience for 25 percent of Minnesota high school students each year.
- Expand summer workshops for teachers to enable every Minnesota teacher to attend a month-long mathematics workshop at least once every three or four summers.
- Increase high school graduation requirements to three years of science and three years of mathematics.
- Require all high school students who plan to attend a postsecondary educational institution to take four years of science and four years of mathematics.
- Set standards for state and local assessment that ensure significant use of long answers, essays, open-ended problems, and calculators.

Planning Issues

- Minnesota needs a comprehensive approach to mathematics and science education covering the entire length of the educational pipeline.
- Minnesota needs to collect better demographic and performance data on mathematics and science education.
- Because Minnesota licenses twice as many elementary teachers as it needs each year, it would be both feasible and beneficial to increase the mathematics requirement for elementary teachers to conform with national MAA and NCTM standards.

- Many elementary teachers in Minnesota need to improve their understanding of the content areas of mathematics where ethnic achievement gaps are greatest.
- Since virtually no minority students are preparing to teach mathematics in Minnesota colleges and universities, vigorous out-of-state recruitment will be the only way to provide a sufficient number of minority mathematics teachers for the urban districts where minority student enrollment is approaching 50%.
- 70% of faculty in Minnesota colleges and universities will retire in the next 10 years, just as the number of college students begins to increase. It will take a major effort to attract and educate a sufficient number of students to replace these mathematicians.
- A multi-dimensional approach to assessment should be the focus of any future plans. Good testing is expensive: it will require communication, creativity, and time.

Achieving Equity

A critically important goal of U.S. mathematics is to attract more women and minorities to pursue degrees in the mathematical sciences. Compared to national averages, Minnesota has an especially poor record in this regard:

- Participation ratios in college-preparatory mathematics of Black and American Indian students in the Minneapolis public schools have increased significantly in the last decade, but still lag behind proportional levels. Fewer than 500 Minnesota Blacks, Hispanics, and American Indians are enrolled in Algebra II—less than 2% of the total enrollment.
- Although Minnesota's drop-out rate both for white students and for students of color is lower than the national average, Minnesota's Blacks drop out at twice the national rate of Black students.
- Women enrolled in Minnesota colleges and universities pursue mathematics degrees at rates that are well below national averages.
- No Blacks, American Indians, or Hispanics earned a mathematics degree from Minnesota private colleges in 1987; no minority student received a secondary school mathematics teaching degree from any Minnesota institution in 1987-88.
- Graduate school enrollment of Black students in Minnesota—in all fields—is half of what it was ten years ago.

Minnesota has a reputation of being a "lily-white" state, not subject to the special problems of minority urban ghettos. Nevertheless, next year a majority of public school children in Minneapolis—as in most of our nation's largest cities—will be minorities. However in some other respects the profile of minority students in Minnesota is not typical of the nation:

- 94% of Minnesota's Blacks live in the Twin Cities metropolitan area.
- Half of Minnesota's Asian population is Hmong.

Paradoxically, Minnesota's slight lead over national averages in pre-college tests is due to superior performance of its minority students as compared to their peers nationwide. Minnesota white students average the same as white students nationally.

- Only 35% of high school seniors are enrolled in advanced mathematics courses.
- Minnesota's total secondary student enrollment in mathematics (72%) is below the national average of 82%.

Ninth grade algebra is the critical filter for educational and career opportunities. Its importance is magnified by the sequential nature of mathematics together with the elective status of subsequent college-preparatory mathematics courses.

- The gap between majority students and Black and American Indian students is smallest in basic arithmetic skills and greatest in higher order problem solving.
- The areas of the school mathematics curriculum where the ethnic achievement gap is greatest correspond precisely to areas of greatest weakness of teachers' own competence.
- Minority students tend to be over-represented in remedial school mathematics and hence they tend to be deprived of the best opportunities to learn.

Achieving Excellence

Teacher and parent expectations for success in mathematics greatly influence student achievement.

- For some students, competitive classroom situations may contribute to mathematics anxiety and avoidance.
- Extra help provided in advance of regular instruction to ensure students' success is more effective than remedial efforts undertaken after students fail.
- Tracking of low-achieving students into low-level classes that focus only on computation results only in lower achievement in the future.

Mathematics education at all levels must offer special opportunities for highly talented students, both to enable these students to achieve their full potential and to ensure a sufficient number of highly educated mathematicians, scientists, and engineers.

- Schools do not do as well at identifying mathematically talented female students as they do with male students.
- Mathematics competitions, magnet schools, talented youth programs, summer institutes, industrial internships, and undergraduate research experiences are among the many effective programs provided by Minnesota institutions for students with special talent in mathematics.
- Residential high schools for mathematics and science—which now exist in eight states—offer many benefits that cannot be duplicated by other special programs. Chief among these is the benefit of day-to-day interaction with peers and with teachers who are specially selected to offer a suitable challenging program of study.
- Energetic efforts to improve the educational climate for young girls who exhibit special talent in mathematics yield performance and persistence by girls that is comparable to that of boys.

It is clear from the record that there is considerable need for expanded opportunities to enhance the mathematical education of Minnesota's talented youth.

Teachers and Schools

Annual demand for elementary school teachers in Minnesota will be relatively constant for most of the next decade, appropriately 900-1000 per year. In recent years, approximately 3000 new teachers are granted elementary licensure in Minnesota each year—half being educated in Minnesota, half elsewhere.

For most of the next decade, average annual demand for secondary mathematics teachers in Minnesota will average between 250 and 400. The number of mathematics teaching graduates has doubled in the past five years. The total annual number of newly licensed mathematics teachers is approximately 400.

This increased supply should be sufficient for Minnesota's needs through the 1990s. However, there are some unsettling signals:

- Many elementary school teachers whose mathematical knowledge is deficient believe themselves to be competent because their view of mathematics is nothing more than computation.
- Recent bumping of younger (and better-prepared) teachers by older teachers as school districts cut staff in response to enrollment declines has seriously weakened the qualifications of mathematics teachers in Minnesota.

Minnesota's minimum standards for secondary mathematics teachers are significantly weaker than nationally recommended norms. However, most Minnesota programs for prospective secondary mathematics teachers conform closely to national recommendations. Minnesota's requirement in mathematics for elementary school licensure is totally inadequate. Most Minnesota college and university requirements are not much better, specifying less than half of the credit hours recommended in the 1983 national standards for preparation of elementary school teachers.

In a time of rapid change in mathematics and in mathematics education, continued professional development of teachers is vitally important. Minnesota support in this regard is spotty:

- Opportunities for continued professional development of mathematics teachers are extremely uneven. In the Twin Cities metropolitan area they are superb—perhaps as good as anywhere in the United States. In rural districts they are very inadequate.
- District-level mathematical leadership for staff and curriculum development in Minnesota is relatively weak at a time when national leaders and many other states are moving to strengthen coordination and planning to meet new national goals for education.

Variations in school district size, resources, and demographic characteristics influences greatly the quality of mathematics education:

- In the 1989 state assessment of mathematics, students in the three largest urban districts performed significantly less well than did students from suburban and rural districts.
- Rural school districts suffer from declining enrollments. Many have only half as many students now as at their peaks 15-20 years ago.
- Fewer than 2 percent of the school districts serving 0.3 percent of the students in the state do not offer a full four-year college preparatory mathematics sequence.
- Teachers in small rural districts are often stretched too thin. It is not uncommon for one individual to teach five or six different courses, coordinate computer development, and coach two or three sports. Mathematics instruction suffers under such circumstances.
- Parental and public pressure to offer calculus in high school is often unwarranted and unwise: it forces smaller districts to support unnecessary frills at the expense of solid preparation for more students.

Applying Mathematics

Mathematical models are used extensively in Minnesota business, industry, and government, not only in "high-tech" industries, but also in such sectors as insurance, medical research, transportation, construction, and manufacturing. Most research and development applications of mathematics involve measurement, modelling, and maximization—Minnesota's *other* "3M."

In addition, statistics used by many Minnesota companies for quality control of manufacturing processes, for estimation of insurance risks, and for economic forecasts. "We could do a lot better job if we had people who understand statistics," reported one company supervisor.

Modelling is used by Minnesota industry to design consumer and industrial products ranging from fan blades to heart valves, from quiet antennas to digital lenses:

- Mayo Clinic uses fractual geometry to analyze medical images.
- Unisys used partial differential equations to develop a quiet antenna—one that does not re-radiate the signals it receives.
- Minnesota medical researchers diagnose heart problems by careful inference from the shape of the wave form of the sound of the heart.
- Mathematicians at the University of Minnesota's Institute for Mathematics and Its Applications (IMA) helped engineers at Honeywell design a digital lens that can do for optics what digital recordings have done for music.
- 3M uses the mathematical theory of splines—curves formed from polynomial pieces fitted together smoothly—to design computer programs to make digital pictures look natural, rather than looking like they were produced on a computer.
- The Minnesota Department of Transportation uses a mathematical model of traffic patterns and road wear to develop a management model for highway and bridge repair.
- The Metropolitan Transit Commission uses a computer-based combinatorial model to schedule bus routes to minimize the number of "deadhead" miles traversed before, between, and after revenue-producing runs.

Student interest in mathematics can be enhanced by introducing students to exciting topics from contemporary applications and research. Minnesota institutions offer a specially rich resource of such topics. Yet by and large, mathematics is taught in Minnesota without any obvious connections to the extensive practice of mathematics among Minnesota employers.

Mathematics teachers—especially college and university teachers—should popularize mathematics by presenting material at several levels of mastery: appreciation, description, procedure. Many newer topics in mathematics could be introduced in high school and college courses as a way to show students that mathematics is an exciting, active field. Examples include:

Factoring large numbers
Fractal geometry
Chaos
Non-linear dynamical systems
Control theory
Computer graphics
Knots
Combinatorics
Three- and four-dimensional geometry
Computational complexity
Iteration and recursion

Technical and Community Colleges

Minnesota's technical colleges—formerly called vocational-technical institutes—are unique among the state's higher educational institutions in employing instructors from industry. Although these instructors can draw on many years of industrial experience, many who teach mathematics have little background in mathematics or experience in teaching. Traditionally, three types of mathematics courses have been common at the technical colleges: remedial or tutorial; "related" mathematics covering procedures for very special areas such as diesel mechanics or real estate; and "core" mathematics covering high school topics.

As part of the present restructuring of the technical colleges, "related math" courses will be reduced in favor of core courses (e.g., in algebra). Moreover, students will for the first time be expected to pass core mathematics courses to receive certification in various programs of study. However, high-school level core mathematics courses are often taught in technical colleges by individuals with no degree or teaching certificate in mathematics.

In recent years, the mathematics curriculum at community colleges has migrated away from a college transfer curriculum to a vocational curriculum more like that of the technical colleges. Consequently, the proportion of students who begin in community colleges and who subsequently finish a four-year degree is much lower than it was in former years.

In general, there is little evidence of innovation in curriculum or pedagogy in either the community or the technical colleges:

- Very few mathematics courses in the community or technical colleges use computers, even though computers are widely used in other programs (e.g., business, computer science) at those same institutions.
- Both technical and community colleges require a state-mandated mathematics placement test. The technical college test emphasizes very basic paper-and-pencil arithmetic skills.
- Calculator arithmetic is a very common topic in various courses in both technical and community colleges.
- In most technical and engineering mathematics courses in community colleges, the predominant teaching technique being used is the traditional classroom lecture approach.
- The algebra-calculus sequence in the technical and engineering tracks of community colleges is very standard, much like the same courses in four-year colleges and universities.

Industry Needs

To maintain Minnesota's 1985 level of mathematics, science, and engineering degrees throughout the mid-1990s will require an increase of at least 30 percent in such degrees over 1985 levels.

Such a sizable increase will be virtually impossible to achieve without significant infusion of women and minorities. Even if Minnesota's high technology industrial growth rate is zero in the coming decade, Minnesota will be more than 5,000 engineers short of current need. If the growth rate is just 5 percent during the 1990s, the science and engineering shortfall will increase by another 20,000.

- Minnesota high technology companies employ over 250,000 people, but the U.S. has fallen behind in international competitiveness in the high technology sector.
- Minnesota ranks 14th among the fifty states in per capital employment of engineers, but only 34th in per capita graduation of new engineers.
- Minnesota's capacity to provide for its own needs for scientists and engineers is dwindling, not increasing.
- A large percentage of engineers who work for large companies in Minnesota—in some cases as high as 50 percent—will retire in the next decade.

To meet national and state needs, school mathematics must encourage all students to use technology; must prepare students for employment in science and engineering; and must ensure effective acquisition of problem-solving skills. If Minnesota expects to be competitive in the future—to avoid repetition of the recent losses of 20,000 jobs in the high technology industry—it will have to assure dramatically increased numbers of scientists, technicians, mathematicians, and engineers.

Assessment

Contrasting headlines dominate news of mathematics assessment results in Minnesota. Minnesota students continue to show a performance advantage over national norms, and performance of Minnesota students on statewide assessments has improved consistently at all grade levels and in nearly all objectives.

Nonetheless, both Minnesota and national scores on national college entrance exams have declined regularly for the last 15 years. During this period, the margin of Minnesota's advantage over national averages has narrowed.

It would be both foolish and impossible to draw conclusions about what students know in mathematics by relying on Minnesota's local district testing programs:

- Too many locally developed assessment tests merely test shopkeeper mathematics using antiquated methods with questionable reliability. The best assessment tools would assess many aspects of what the *NCTM Standards* call "mathematical power."
- Present assessment practice in Minnesota is based on a multiple-choice methodology that is inadequate to fully assess what students know about mathematics and is inconsistent with the NCTM standards for evaluation.
- The absence of calculators from assessment tests impacts the curriculum negatively by sending a wrong message at the wrong time.
- Students who are not accustomed to open-ended problems will need to develop the mathematical disposition appropriate to these new forms of assessment. Old habits will not die easily.
- Many teachers are uncertain of their ability to provide appropriate guidance to help students learn how to approach open-ended problems.

Curriculum reform in mathematics must occur in concert with assessment reforms. If tests lead curriculum, there is no validity in the results, while if the curriculum leads the test, there is no value in the results.

APPENDIX D

Mathematics Contacts

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS (NCTM)

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MINNESOTA COUNCIL OF TEACHERS OF MATHEMATICS (MCTM)

Executive Director, David Dye
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MINNESOTA CURRICULUM SERVICES CENTER (MCSC)

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APPENDIX E

CORRELATION CHART

This chart should be of use in comparing the location of the 1988 Chapter 3 MLOs to the <u>new</u> location in the 1990 Chapter 3 MLOs. The 1988 MLOs are listed by the coding on the left side. There is an "X" where they are found in the new organization in the 1990 MLOs as listed across the top.

Math Program Outcomes	A	B	C	D	E	F	G	H	I	J
1988 MLO										
I.A			X							
I.B					X					
I.C					X					
I.D					X					
I.E					X					
I.F					X					
I.G						X				
I.H					X					
I.I						X				
I.J						X				
I.K					X					
II.A						X				
II.B						X				
II.C							X			
II.D						X				
II.E							X			
II.F							X			
III.A					X					
III.B					X					
III.C					X					
III.D					X					
III.E					X					
III.F					X					
III.G					X					
IV.A								X		
IV.B								X		
IV.C								X		
IV.D								X		
IV.E								X		
IV.F								X		
IV.G									X	
V.A						X				
V.B						X				
V.C						X				
V.D						X				
V.E							X			
V.F						X				
V.G							X			
V.H							X			
V.I						X				

2 OF 3

Math Program										
Outcomes	A	B	C	D	E	F	G	H	I	J
1988 MLO										
VI.A										X
VI.B										X
VI.C										X
VI.D										X
VI.E									X	
VI.E									X	
VI.F									X	
VI.G										X
VI.H										X
VII.A										X
VII.B									X	
VII.C									X	
VII.D									X	
VII.E									X	
VII.F									X	
VIII.A										X
VIII.B										X
VIII.C										X
VIII.D										X
IX.A	X									
IX.B	X									
IX.C	X									
IX.D	X									
IX.E	X									
IX.F	X									
IX.G	X									
IX.H	X									
X.A			X							
X.B		X								
X.C				X						
XI.A							X			
XI.B									X	
XI.C									X	
XI.D						X				
XI.E							X			
XII.A		X								
XII.B		X								
XIII.A	X									
XIII.B	X									
XIII.C	X									
XIII.D	X									

Math Program										
Outcomes	A	B	C	D	E	F	G	H	I	J
1988 MLO										
XIII.E			X							
XIII.F		X								
XIII.G		X								
XIII.H		X								
XIV.A				X						
XIV.B				X						
XIV.C				X						
XIV.D				X						
XIV.E				X						
XIV.F				X						
XIV.G				X						
XIV.H				X						
XIV.I				X						
XV.A										X
XV.B										X
XV.C		X								
XV.D				X						

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